# ODE

## Sample Midterm 2 Math 3331 (Summer 2014) June 24, 2014

20 points 1. Use the Laplace Transform to find the solution of the following initial-value problems

a. $y'' + y = \cos 2t, \quad y(0) = 0, \quad y'(0) = 1.$

*Solution* Let $Y(s) = \mathcal{L}(y(t))$. Then

$$\mathcal{L}(y'' + y) = \mathcal{L}\{\cos 2t\} \quad \Rightarrow \quad \mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}\{\cos 2t\}$$
$$\Rightarrow \quad \big(s^2 Y - sy(0) - y'(0)\big) + Y = \frac{s}{s^2+4}$$
$$\text{I.Cs.} \quad \Rightarrow \quad (s^2+1)Y - 1 = \frac{s}{s^2+4} \quad \Rightarrow \quad Y(s) = \frac{s^2+s+4}{(s^2+1)(s^2+4)}$$

Note that the partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{s^2+s+4}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

Then

$$y(t) = \mathcal{L}^{-1}(Y(s))(t) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)$$
$$= \frac{1}{3}\cos t + \sin t - \frac{1}{3}\cos 2t$$

b. $y'' - y = e^t, \quad y(0) = 0, \quad y'(0) = 0.$

*Solution* Let $Y(s) = \mathcal{L}(y(t))$. Then

$$\mathcal{L}(y'' - y) = \mathcal{L}\{e^t\} \quad \Rightarrow \quad \mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}\{e^t\}$$
$$\Rightarrow \quad \big(s^2 Y - sy(0) - y'(0)\big) - Y = \frac{1}{s-1}$$
$$\text{I.Cs.} \quad \Rightarrow \quad (s^2-1)Y = \frac{1}{s-1} \quad \Rightarrow \quad Y(s) = \frac{1}{(s+1)(s-1)^2}$$

Note that the partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{1}{(s+1)(s-1)^2} = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}$$

Then

$$y(t) = \mathcal{L}^{-1}(Y(s))(t) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)$$
$$= \frac{1}{4}e^{-t} - \frac{1}{4}e^t + \frac{1}{2}te^t$$

20 points 2. Consider the initial value problem

$$x' = -x + t, \quad 0 \leq t \leq 1, \quad x(0) = 0.5. \tag{1}$$

Use the Euler, RK2 and RK4 methods to approximate the value of $x(1)$ for a step size $h = 0.5$ and compute the error of your numerical solution.

*Solution* We have $t_0 = 0$, $x_0 = 0.5$, and $f(t, y) = t - x$. Thus, the first step of Euler's method is completed as follows

$$x_1 = x_0 + hf(t_0, x_0) = 0.5 + 0.5(0 - 0.5) = 0.25$$
$$t_1 = t_0 + h = 0 + 0.5 = 0.5$$

The second step follows

$$x_2 = x_1 + hf(t_1, x_1) = 0.25 + 0.5 * (0.5 - 0.25) = 0.375$$
$$t_2 = t_1 + h = 0.5 + 0.5 = 1$$

The first step of RK2 method follows. First we compute the slopes

$$s_1 = f(t_0, x_0) = f(0, 0.5) = 0 - 0.5 = -0.5$$
$$s_2 = f(t_0 + h, x_0 + hs_1) = f(0.5, 0.25) = 0.5 - 0.25 = 0.25$$

Youn can now update $x$ and $t$

$$x_1 = x_0 + h\frac{1}{2}(s_1 + s_2) = 0.5 + 0.5\frac{1}{2}(-0.5 + 0.25) = 0.4375$$
$$t_1 = t_0 + h = 0 + 0.5 = 0.5$$

The second iteration begins with computing the slopes

$$s_1 = f(t_1, x_1) = f(0.5, 0.4375) = 0.5 - 0.4375 = 0.0625$$
$$s_2 = f(t_1 + h, x_1 + hs_1) = f(1, 0.46875) = 1 - 0.46875 = 0.53125$$

You can now update $x$ and $t$

$$x_2 = x_1 + h\frac{1}{2}(s_1 + s_2) = 0.4375 + 0.5\frac{1}{2}(0.0625 + 0.53125) = 0.5859375$$
$$t_2 = t_1 + h = 0.5 + 0.5 = 1$$

The first step of RK4 method follows. First we compute the four slopes

$$s_1 = f(t_0, x_0) = f(0, 0.5) = -0.5 + 0 = -0.5$$
$$s_2 = f(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}s_1) = f(0.25, 0.375) = 0.25 - 0.375 = -0.125$$
$$s_3 = f(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}s_2) = f(0.25, 0.46875) = 0.25 - 0.46875 = -0.21875$$
$$s_4 = f(t_0 + h, x_0 + hs_3) = f(0.5, 0.390625) = 0.5 - 0.390625 = 0.109375$$

Youn can now update $x$ and $t$

$$\begin{aligned} x_1 &= x_0 + h\frac{1}{6}(s_1 + 2(s_2 + s_3) + s_4) \\ &= 0.5 + 0.5\frac{1}{6}(-0.5 + 2(-0.125 - 0.21875) + 0.109375) = 0.41015625 \\ t_1 &= t_0 + h = 0 + 0.5 = 0.5 \end{aligned}$$

The second iteration begins with computing the slopes

$$\begin{aligned} s_1 &= f(t_1, x_1) = f(0.5, 0.41015625) = 0.5 - 0.41015625 = 0.08984375 \\ s_2 &= f(t_1 + \frac{h}{2}, x_1 + \frac{h}{2}s_1) = f(0.75, 0.432617188) = 0.75 - 0.432617188 = 0.317382812 \\ s_3 &= f(t_1 + \frac{h}{2}, x_1 + \frac{h}{2}s_2) = f(0.75, 0.489501953) = 0.75 - 0.489501953 = 0.260498047 \\ s_4 &= f(t_1 + h, x_1 + hs_3) = f(1, 0.540405274) = 1 - 0.540405274 = 0.459594726 \end{aligned}$$

You can now update $x$ and $t$

$$\begin{aligned} x_2 &= x_1 + h\frac{1}{6}(s_1 + 2(s_2 + s_3) + s_4) \\ &= 0.41015625 + 0.5\frac{1}{6}(0.08984375 + 2(0.317382812 + 0.260498047) + 0.459594726) = 0.552256266 \\ t_2 &= t_1 + h = 0.5 + 0.5 = 1 \end{aligned}$$

The equation is linear and we can find its solution $x(t) = \frac{3}{2}e^{-t} + t - 1$ (note that $x(t) = x_h(t) + x_p(t)$ with $x_h(t) = ce^{-t}$ and $x_p(t) = t - 1$). We can compute the true values: $x(0.5) = \frac{3}{2}e^{-0.5} + 0.5 - 1 = 0.40979599$ and $x(1) = \frac{3}{2}e^{-1} = 0.551819162$ We can complete the following table

| | time | approx. | true value | error |
|---|---|---|---|---|
| Euler | 0.5 | 0.25 | 0.40979599 | 0.15979599 |
| RK2 | 0.5 | 0.4375 | 0.40979599 | 0.02770401 |
| RK4 | 0.5 | 0.41015625 | 0.40979599 | 0.00036026 |
| Euler | 1.0 | 0.375 | 0.551819162 | 0.176819162 |
| RK2 | 1.0 | 0.5859375 | 0.551819162 | 0.034118338 |
| RK4 | 1.0 | 0.552256266 | 0.551819162 | 0.000437104 |

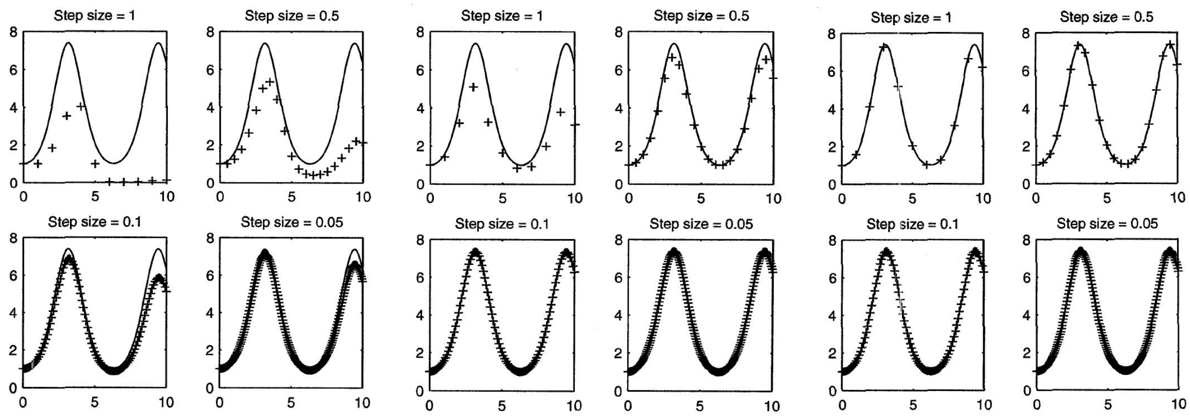
20 points

3. Consider the initial value problem

$$x' = x \sin t, \quad t \geq 0, \quad x(0) = 1. \tag{2}$$

The equation is separable and the solution is $x(t) = e^{1-\cos t}$. The Euler method, RK2 and RK4 methods, with step sizes $h = 1$, 0.5, 0.1 and 0.05 produce the following results. Indicate each graph (1,2,3) by its corresponding numerical method and explain your answer.

(1) (2) (3)

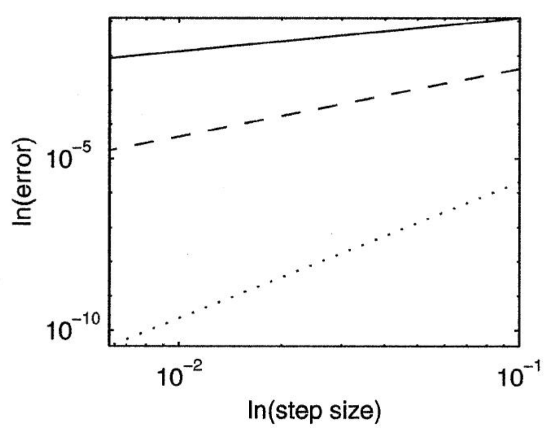



*Solution* Euler's method , with step sizes $h = 1$, 0.5, 0.1 and 0.05, produces the results shown in Graph 1, RK2 method does a little better producing the results in Graph 2, RK4 method is the most accurate, producing the results in Graph 3.

20 points 4. Consider the initial value problem

$$x' = x, \quad 0 \le t \le 1, \quad x(0) = 1. \tag{3}$$

The equation is separable and the solution is $x(t) = e^t$. We used the Euler method, RK2 and RK4 methods to compute the value of $x(1)$ and constructed a plot of the logarithm of the error versus the logarithm of the step size for each numerical method. The slope of the solid line is 0.9716, the slope of the dashed line is 1.9755, and the slope of the dotted line is 3.9730. Indicate each line by its corresponding numerical method and explain your answer.




*Solution* Euler's method produces the results shown in the solid line, as the slope of the solid line is 0.9716, which is closed to 1, consistent with the fact that the Euler method is a first order algorithm. RK2 method produces the results shown in the dashed line as the slope of the dashed line is 1.9755, which is close to 2, consistent with the fact that RK2 is a second order method. Finally, RK4 method produces the results shown in the dotted line as the slope of the dotted line is 3.9730, which is close to 4, consistent with the fact that RK4 is a fourth order method.

20 points 5. Write each initial value problems as a system of the first-order equations using vector notation.

$$a. \qquad x'' + \delta x' - x + x^3 = \gamma \cos \omega t, \quad x(0) = x_0,\ x'(0) = v_0$$

$$b. \qquad x'' + \mu(x^2 - 1)x' + x = 0, \quad x(0) = x_0,\ x'(0) = v_0$$

*Solution* a. With

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

we have

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -\delta u_2 + u_1 - u_1^3 + \gamma \cos \omega t, \end{aligned}$$

with initial conditions

$$u(0) = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

b. With

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

we have

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -\mu(u_1^2 - 1)u_2 - u_1, \end{aligned}$$

with initial conditions

$$u(0) = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$