ODE

Sample Midterm 3 Math 3331 (Summer 2014) July 2, 2014

50 points 1. Find the solution of the initial-value problem

$$\begin{aligned} x' &= -3x \\ y' &= -5x + 6y - 4z \\ z' &= -5x + 2y \end{aligned}$$

with $x(0) = -1$, $y(0) = 0$ and $z(0) = 1$.

Solution In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{pmatrix}$ are

$$\begin{aligned} &\lambda_1 = 4, \quad \lambda_2 = -3, \quad \lambda_3 = 2 \\ &v_1 = (0,2,1)^T, \quad v_2 = (1,1,1)^T, \quad v_3 = (0,1,1)^T. \end{aligned}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{4t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

If $x(0) = -1$, $y(0) = 0$ and $z(0) = 1$, then

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

We find that $c_1 = -1$, $c_2 = -1$, and $c_3 = 3$. Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = -e^{4t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-3t} \\ -2e^{4t} - e^{-3t} + 3e^{2t} \\ -e^{4t} - e^{-3t} + 3e^{2t} \end{pmatrix}$$

50 points 2. Find the solution of the initial-value problem

$$\begin{aligned} x' &= 6x - 4z \\ y' &= 8x - 2y \\ z' &= 8x - 2z \end{aligned}$$

with $x(0) = -2$, $y(0) = -1$ and $z(0) = 0$.

Solution In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{pmatrix}$ are

$$\begin{aligned} &\lambda_1 = -2, \quad \lambda_2 = 2 + 4i, \quad \lambda_3 = 2 - 4i \\ &v_1 = (0, 1, 0)^T, \quad v_2 = (1 + i, 2, 2)^T, \quad v_3 = (1 - i, 2, 2)^T. \end{aligned}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \cos 4t - \sin 4t \\ 2\cos 4t \\ 2\cos 4t \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} \cos 4t + \sin 4t \\ 2\sin 4t \\ 2\sin 4t \end{pmatrix}$$

If $x(0) = -2$, $y(0) = -1$ and $z(0) = 0$, then

$$\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We find that $c_1 = -1$, $c_2 = 0$, and $c_3 = -2$. Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = -e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2e^{2t} \begin{pmatrix} \cos 4t + \sin 4t \\ 2\sin 4t \\ 2\sin 4t \end{pmatrix} = \begin{pmatrix} -2e^{2t}(\cos 4t + \sin 4t) \\ -e^{-2t} - 4e^{2t}\sin 4t \\ -4e^{2t}\sin 4t \end{pmatrix}$$

20 points 3. (**BONUS PROBLEM**) Classify the equilibrium point of the system $y' = Ay$. Sketch the phase portrait by hand.

$$(1)\ A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix} \qquad (2)\ A = \begin{pmatrix} 2 & 1 \\ -10 & -5 \end{pmatrix}$$

Solution (1) If

$$A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix}$$

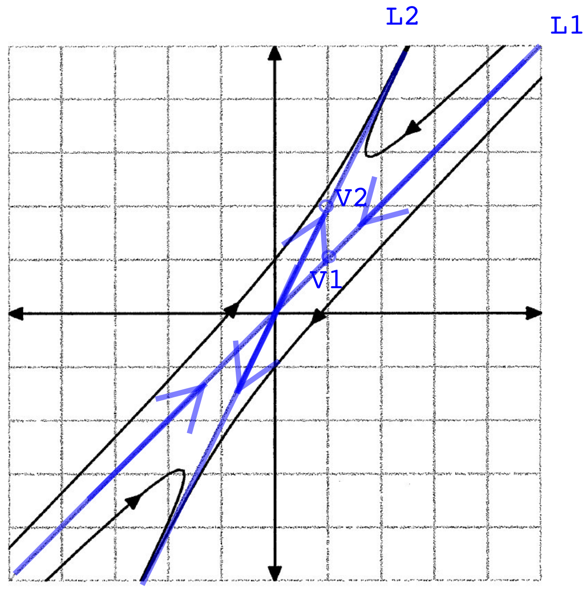
then the trace is $T = -5$ and the determinant is $D = -14 < 0$. Hence, the equilibrium point at the origin is a saddle. The eigen-pairs of $A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix}$ are

$$\lambda_1 = -7, \quad v_1 = (1,1)^T,$$
$$\lambda_2 = 2, \quad v_2 = (1,2)^T.$$

The general solution is

$$y(t) = c_1 e^{-7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solutions approach the halfline generated by $c_2(1,2)^T$ as they move forward in time, but they approach the halfline generated by $c_1(1,1)^T$ as they move backward in time. A hand sketch follows



(2) If

$$A = \begin{pmatrix} 2 & 1 \\ -10 & -5 \end{pmatrix}$$

then the trace is $T = -3 < 0$ and the determinant is $D = 0$. Thus, this degenerate case lies on the horizontal axis in the trace-determinant plane, separating the saddles from the nodal sinks. The eigen-pairs of $A = \begin{pmatrix} 2 & 1 \\ -10 & -5 \end{pmatrix}$ are

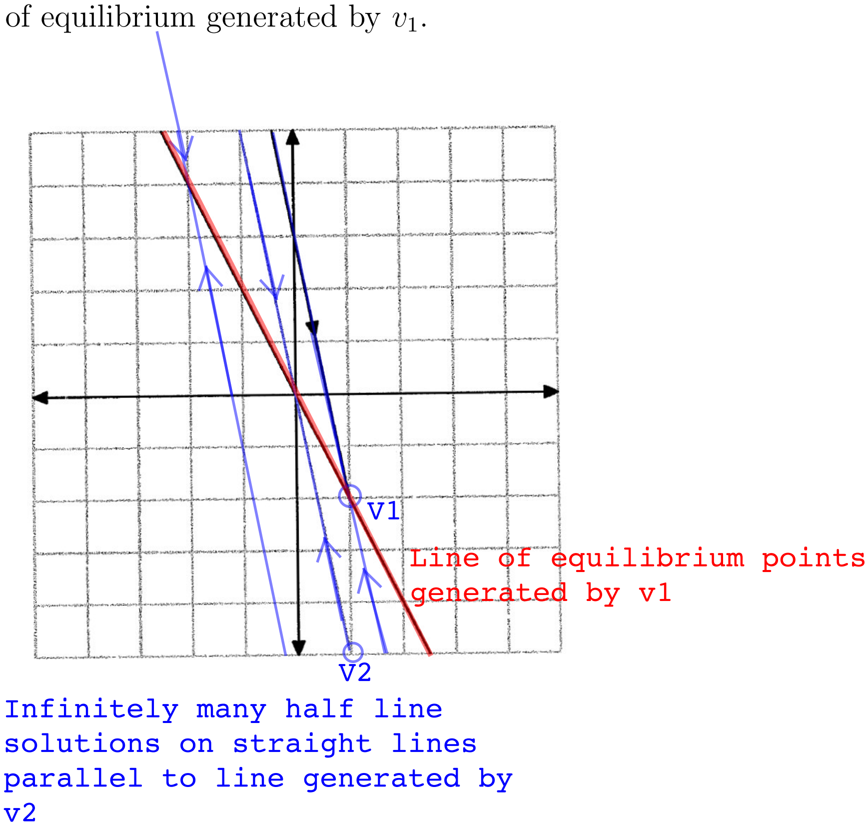
$$\lambda_1 = 0, \quad v_1 = (1,-2)^T,$$
$$\lambda_2 = -3, \quad v_2 = (1,-5)^T.$$

The general solution is

$$y(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

Each Solution in this family is the sum of a fixed multiple of $(1,-2)^T$ and a decaying multiple of $(1,-5)^T$. Thus, as $t \to \infty$, solutions move in lines parallel to $(1,-5)^T$, decaying into the line of equilibrium generated by v_1.

A hand sketch follows



Eigenvalues and eigenvectors of matrices

- The eigen-pairs of $A = \begin{pmatrix} 2 & 1 \\ -10 & -5 \end{pmatrix}$ are

$$\lambda_1 = 0, \quad v_1 = (1, -2)^T,$$
$$\lambda_2 = -3, \quad v_2 = (1, -5)^T.$$

- The eigen-pairs of $A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix}$ are

$$\lambda_1 = -7, \quad v_1 = (1, 1)^T,$$
$$\lambda_2 = 2, \quad v_2 = (1, 2)^T.$$

- The eigen-pairs of $A = \begin{pmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{pmatrix}$ are

$$\lambda_1 = -2, \quad v_1 = (0, 1, 0)^T,$$
$$\lambda_2 = 2 + 4i, \quad v_2 = (1 + i, 2, 2)^T,$$
$$\lambda_3 = 2 - 4i, \quad v_3 = (1 - i, 2, 2)^T.$$

- The eigen-pairs of $A = \begin{pmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{pmatrix}$ are

$$\lambda_1 = 4, \quad v_1 = (0, 2, 1)^T,$$
$$\lambda_2 = -3, \quad v_2 = (1, 1, 1)^T,$$
$$\lambda_3 = 2, \quad v_3 = (0, 1, 1)^T.$$