

Math 3331 Differential Equations

2.2 Solutions to Separable Equations

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2.2 Solutions to Separable Equations

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 - General Method
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- Applications
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- Worked out Examples from Exercises:
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 - Find Solutions to IVPs and IODEs: 13, 15, 17, 19
 - Application: 26, 33



The General Method: Separation of Variables

Form: $\frac{dy}{dt} = g(t)f(y)$

Implicit Solution:

$$[1/f(y)]dy = g(t)dt$$

$$\int [1/f(y)]dy = \int g(t)dt \quad (*)$$

or $H(y) = G(t) + C$ where

$$H(y) = \int [1/f(y)]dy$$

$$G(t) = \int g(t)dt$$

Solve (*) for $y \rightarrow$ explicit solution

Note: (*) may have several solutions.
Use IC to choose the right one.



Example

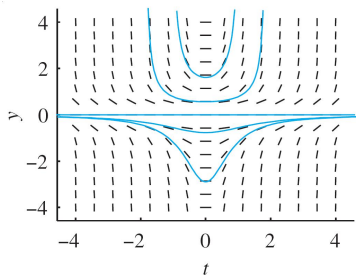
$$\text{Ex.: } \frac{dy}{dt} = ty^2$$

$$(1/y^2)dy = t dt$$

$$\Rightarrow \int (1/y^2)dy = \int t dt$$

$$\Rightarrow -1/y = t^2/2 + C$$

$$\begin{aligned}\Rightarrow y(t) &= -1/(t^2/2 + C) \\ &= -2/(t^2 + 2C)\end{aligned}$$



Example: Exponential Equation

Ex.: Find gen. sol. to $dx/dt = rx$

$$\frac{dx}{x} = r dt \Rightarrow \ln|x| = rt + C$$

$$\Rightarrow |x(t)| = e^{rt+C} = e^C e^{rt}$$

$$x(t) > 0 \Rightarrow x(t) = e^C e^{rt}$$

$$x(t) < 0 \Rightarrow x(t) = -e^C e^{rt}$$

Set $A = e^C$ if $x > 0$, $A = -e^C$ if $x < 0$

$$\Rightarrow x(t) = A e^{rt}$$

with arbitrary constant A (can be 0)

Initial value: $x(0) = A$



Example: IVP

Example: general linear equation with constant coefficients

$$y' = ry + a, \text{ IC: } y(0) = y_0 \quad (r, a, y_0: \text{arbitrary parameters})$$

$$[1/(ry + a)]dy = dt \Rightarrow (\ln |ry + a|)/r = t + C \Rightarrow |ry + a| = e^{rt+rC} = e^{rC}e^{rt}$$

$$\Rightarrow ry + a = Ae^{rt} \quad (A = \pm e^{rC}) \Rightarrow y(t) = (Ae^{rt} - a)/r$$

$$\text{Invoke IC: } y(0) = (A - a)/r = y_0 \Rightarrow A = ry_0 + a \Rightarrow y(t) = (y_0 + a/r)e^{rt} - a/r$$



Implicitly Defined Solutions

Find sols. of $x' = 2tx/(1+x)$
 s.t. $x(0) = 1$, $x(0) = -2$, and
 $x(0) = 0$.

Answer:

$$(1 + 1/x)dx = 2t dt, \quad x \neq 0$$

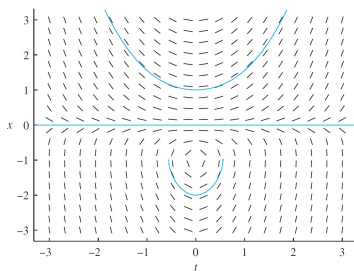
$$\Rightarrow x + \ln(|x|) = t^2 + C$$

(i) For $x(0) = 1$

$$\Rightarrow C = 1$$

$$x + \ln x - 1 = t^2$$

$\Rightarrow x(t)$ implicitly defined.



Exercise 2.2.1

Ex. 1: $y' = xy$

$$\begin{aligned}(1/y)dy = xdx &\Rightarrow \ln |y| = x^2/2 + C \Rightarrow |y| = \exp(x^2/2 + C) = e^C e^{x^2/2} \\ &\Rightarrow y(x) = Ae^{x^2/2}, \quad A = e^C \text{ or } A = -e^C\end{aligned}$$



Exercise 2.2.3

Ex. 3: $y' = e^{x-y}$

$$e^y dy = e^x dx \Rightarrow e^y = e^x + C \Rightarrow y(x) = \ln(e^x + C)$$



Exercise 2.2.5

Ex. 5: $y' = y(x + 1)$

$$(1/y)dy = (x+1)dx \Rightarrow \ln|y| = x^2/2+x+C \Rightarrow |y| = e^C e^{x+x^2/2} \Rightarrow y(x) = Ae^{x+x^2/2}$$



Exercise 2.2.9

$$\text{Ex. 9: } x^2 y' = y \ln y - y' \Rightarrow y' = (y \ln y)/(1 + x^2)$$

$$[1/(y \ln y)] dy = [1/(1 + x^2)] dx \Rightarrow \ln(\ln y) = \arctan x + C$$

$$\Rightarrow y(x) = \exp(e^C e^{\arctan x}) = \exp(D e^{\arctan x}) \quad (D = e^C)$$



Exercise 2.2.11

Ex. 11: $y^3 y' = x + 2y' \Rightarrow y' = x/(y^3 - 2)$

$$(y^3 - 2)dy = x dx \Rightarrow y^4/4 - 2y = x^2/2 + C \Rightarrow \text{implicit sol.: } y^4 - 8y - 2x^2 = D \quad (D = 4C)$$



Exercise 2.2.13

Ex. 13: $y' = y/x$, IC: $y(1) = -2$

General sol.: $(1/y)dy = (1/x)dx \Rightarrow \ln |y| = \ln |x| + C$

$$\Rightarrow |y| = \exp(C + \ln |x|) = e^C e^{\ln |x|} = e^C |x| \Rightarrow y(x) = Ax \quad (A = \pm e^C)$$

Match C to IC: $y(1) = A = -2 \Rightarrow y(x) = -2x$; IoE: $(0, \infty)$



Exercise 2.2.15

Ex. 15: $y' = (\sin x)/y$, IC: $y(\pi/2) = 1$

$$y dy = \sin x dx \Rightarrow y^2/2 = -\cos x + C \Rightarrow y = \pm\sqrt{D - 2\cos x} \quad (D = 2C)$$

$$y(\pi/2) = 1 > 0 \Rightarrow \text{need '+'-sign} \Rightarrow y(\pi/2) = \sqrt{D} = 1 \Rightarrow y(x) = \sqrt{1 - 2\cos x}$$

Find IoE: need $\cos x < 1/2 \Rightarrow \text{IoE: } (\pi/3, 5\pi/3)$



Exercise 2.2.17

Ex. 17: $y' = 1 + y^2$, IC: $y(0) = 1$

$$[1/(1 + y^2)]dy = dt \Rightarrow \arctan y = t + C \Rightarrow y = \tan(t + C) + k\pi \quad (k : \text{integer})$$

$$\text{Since } y(0) = 1 \Rightarrow k = 0 \Rightarrow y(t) = \tan(t + C)$$

$$\text{Invoke IC: } y(0) = \tan C = 1 \Rightarrow C = \pi/4 \Rightarrow y(t) = \tan(t + \pi/4)$$

$$\text{For IoE: need } t + \pi/4 > -\pi/2 \text{ and } t + \pi/4 < \pi/2 \Rightarrow \text{IoE: } (-3\pi/4, \pi/4)$$



Exercise 2.2.19

Ex. 19: $y' = x/y$, $IC_1: y(0) = 1$ and $IC_2: y(0) = -1$

$$y dy = x dx \Rightarrow y^2/2 = x^2/2 + C \Rightarrow y = \pm\sqrt{x^2 + D} \quad (D = 2C)$$

$$IC_1: y(0) = 1 \Rightarrow y(0) = +\sqrt{D} = 1 \Rightarrow y(x) = \sqrt{1 + x^2}$$

$$IC_2: y(0) = -1 \Rightarrow y(0) = -\sqrt{D} = -1 \Rightarrow y(x) = -\sqrt{1 + x^2}$$



Radioactive Decay

$N(t)$: # of radioactive atoms

- Model: $dN/dt \sim -N$

$$\Rightarrow dN/dt = -\lambda N$$

- Solution: $N(t) = N_0 e^{-\lambda t}$
- Half-life:

$$N(t)/N(0) = e^{-\lambda t} = 1/2$$

$$\Rightarrow t = (\ln 2)/\lambda \equiv T_{1/2}$$

- Natural log of ratios:

$$\ln[N_0/N(t)] = \lambda t$$

- Use $\lambda = (1/t) \ln[N_0/N(t)]$ to determine λ from measurement
- Use $t = (1/\lambda) \ln[N_0/N(t)]$ to determine time t^* s.t. $N(t^*) = N^*$ for given N^*



Exercise 2.2.25

Ex. 25: After $t = 4 \text{ hrs}$, 80 mg of a 100 mg sample of Tritium remain. Determine λ and $T_{1/2}$.

Answer:

$$\lambda = (1/4) \ln[100/80] = 0.056/\text{hrs}$$

$$T_{1/2} = (\ln 2)/0.056 = 12.43 \text{ hrs}$$



Exercise 2.2.26

Ex. 26: $T_{1/2} = 6 \text{ hrs}$ for Technetium $99m$. What remains after 9 hrs if $N_0 = 10 \text{ g}$?

Answer:

$$\lambda = (\ln 2)/6 = 0.116/\text{hr}$$

$$\Rightarrow N(9) = 10e^{-0.116 \times 9} = 3.54 \text{ g}$$



Newton's Law of Cooling

$T(t)$: temperature of object

A : surrounding temperature

- Model: $dT/dt \sim A - T$

$$\Rightarrow dT/dt = k(A - T)$$

- Solution (see Example p.5):

$$T(t) = A + e^{-kt}(T_0 - A)$$

- $(T - A)/(T_0 - A) = e^{-kt} \Rightarrow$

$$kt = \ln[(T_0 - A)/(T(t) - A)]$$

$$k = (1/t) \ln[(T_0 - A)/(T(t) - A)]$$

→ determine k

$$t = (1/k) \ln[(T_0 - A)/(T(t) - A)]$$

→ determine t



Exercise 2.2.33

Ex. 33: Dead body found at $t = 0$ (midnight). Temperature 31°C .

1 hr later (1 *am*): Temperature 29°C

Surrounding temperature: $A = 21^{\circ}\text{C}$

Question: When did death (murder) occur?

Answer: $t = 1 \text{ hr}$, $T_0 = 31$, $T(1) = 29$

$$\begin{aligned}\Rightarrow k &= (1/1) \ln[(31 - 21)/(29 - 21)] \\ &= 0.223/\text{hr}\end{aligned}$$

Determine time at which $T = 37$:

$$\begin{aligned}t &= (1/k) \ln[(31 - 21)/(37 - 21)] \\ &= -2.11 \text{ hrs} = -2 \text{ hrs } 7 \text{ min}\end{aligned}$$

\Rightarrow Death occurred at 9 : 53 *pm*

