Math 3331 Differential Equations

2.3 Models of Motion

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math3331

2.3 Models of Motion

- Motion of a Ball Near Surface of the Earth
  - Without Air Resistance
  - With Air Resistance
    - Linear Model
    - Quadratic Model

Motion of a Ball Near Surface of the Earth

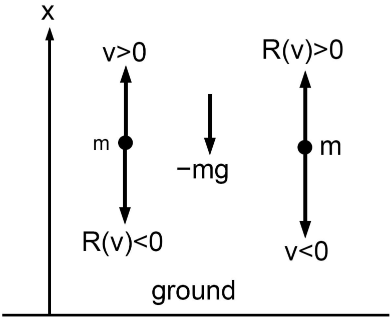
- **Gravity Force:** $\mathrm{F_g} = -\mathrm{mg}$
  - $g = 32\, ft/s^2 = 9.8\, m/s^2$
- **Air Resistance:** $\mathrm{F_{air}} = \mathrm{R(v)}$
  - **Linear Model:**

$$R(v) = -kv$$

    - $[k] =$ mass/time
    - valid for small velocities
  - **Quadratic Model:**

$$R(v) = -k|v|v$$

    - $[k] =$ mass/length
    - valid for larger velocities
- We treat only the linear model.



Solution of the Motion without Air Resistance

$$v' = -g, \quad x' = v$$

$$\Rightarrow \quad v(t) = -gt + v_0$$
$$x(t) = -gt^2/2 + v_0 t + x_0$$

$$\Rightarrow \quad t = (v_0 - v)/g$$
$$x = (v_0^2 - v^2)/(2g) + x_0$$

$$\Rightarrow \quad v^2 = v_0^2 + 2g(x_0 - x)$$

Max Height (if $v_0 > 0$):

$$v = 0 \quad \Rightarrow \quad t_{max} = v_0/g, \quad x_{max} = x_0 + v_0^2/(2g)$$

Ground Hit: $x = 0 \Rightarrow$

$$v_g = -\sqrt{v_0^2 + 2gx_0} \text{ (impact velocity)}$$
$$t_g = \left(v_0 + \sqrt{v_0^2 + 2gx_0}\right)/g$$

Example 1

Example 1:

Ascending balloon, velocity $15\,m/s$. At height $100\,m$ package is dropped. When does package reach ground?

$$g = 9.8\,m/s^2$$

Initial Values:

$$x_0 = 100\,m, \;\; v_0 = 15\,m/s$$

$$\Rightarrow \begin{aligned} t_{max} &= 1.5\,s \\ x_{max} &= 111.5\,m \\ v_g &= 46.7\,m/s \\ t_g &= 6.3\,s \end{aligned}$$

Solution of the Motion with Air Resistance

$$v' = -g - (k/m)v$$

In CN 2.2, Example p.5, we showed:

$$\begin{aligned} & y' = ry + a, \quad y(0) = y_0 \\ \Rightarrow \quad & y(t) = (y_0 + a/r)e^{rt} - a/r \end{aligned}$$

Here: $y = v$, $a = -g$, $r = -k/m$

$$\Rightarrow \quad v(t) = (v_0 + gm/k)e^{-kt/m} - gm/k$$

Integrate this to find x:

$$\begin{aligned} x(t) &= \int_0^t v(t')dt' + x_0 \\ &= \frac{m}{k}(v_0 + gm/k)(1 - e^{-kt/m}) \\ &\quad -(gm/k)t + x_0 \end{aligned}$$

Terminal Velocity:

$$v_{term} \equiv \lim_{t\to\infty} v(t) = -gm/k$$

Example 2

Example 2: (see text, Example 3.8)
$m = 2\,kg$, $k = 4\,kg/m$ $(g = 9.8\,m/s^2)$
Initial values: $x_0 = 250\,m$, $v_0 = 0$

Question:
Time of ground hit? Impact velocity?

Answer:
Ground hit $\rightarrow$ equation for $t = t_g$:

$$\begin{aligned} 0 &= g(m/k)^2(1 - e^{-kt/m}) - (gm/k)t + x_0 \\ &= 2.45(1 - e^{-2t}) - 4.9t + 250 \\ &= 252.45 - 2.45e^{-2t} - 4.9t \end{aligned}$$

Equation solver $\rightarrow t_g \approx 51.52\,s$

Impact velocity:

$$v_g = v(t_g) = 4.9(e^{-2t_g} - 1) \approx -4,9\,m/s$$

Without air resistance:

$$\begin{aligned} t_g &= \sqrt{2x_0/g} \approx 7.14\,s \\ v_g &= -\sqrt{2gx_0} \approx -44.3\,m/s \end{aligned}$$

Graphs for Example 2