Math 3331 Differential Equations

2.4 Linear Equations

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2.4 Linear Equations

- Homogeneous Equations
- Nonhomogeneous Equations
  - Integrating Factor
  - Variation of Parameter
- Worked out Examples from Exercises:
  - Find General Solutions: 3, 5, 7, 11, 12, 33, 35
  - Find Solutions to IVPs and IoEs: 15, 17, 19, 21, 37, 39

Linear Equations: General Form

General Form:

$$x' = a(t)x + f(t) \qquad (1)$$

If $f(t) = 0$, (1) is called **homogeneous**:

$$x' = a(t)x$$

If $f(t) \neq 0$, (1) is called **nonhomogeneous**

Examples of Linear Equations

Examples of linear equations:

$x' = \sin(t)x$	homogenous, $a(t) = \sin t$
$x' = x/t$	homogenous, $a(t) = 1/t$
$y' = e^t y + \cos t$	nonhomogeneous, $a(t) = e^t$, $f(t) = \cos t$
$x' = 3tx + t^2$	nonhomogeneous, $a(t) = 3t$, $f(t) = t^2$

Examples of Non-Linear Equations

Examples of non-linear equations:

$$\begin{aligned} x' &= t \sin x \\ y' &= 1/y \\ y' &= 1 - y^2 \\ u' &= e^{-u} + \cos x \end{aligned}$$

Homogeneous Equations

HE: $x' = a(t)x$ (2)

$$\Rightarrow \frac{dx}{x} = a(t)dt$$
$$\Rightarrow \ln|x| = \int a(t)dt + D$$
$$\Rightarrow |x(t)| = \exp(D + \int a(t)dt)$$
$$= e^D \exp\Big(\int a(t)dt\Big)$$

$\Rightarrow$ **General Solution**:

$$x(t) = C\exp\Big(\int a(t)dt\Big) \quad (3)$$

where $C = \pm e^D$ (any value)

Equivalent form of gen. sol.:

$$x(t) = x_0 \exp\Big(\int_{t_0}^{t} a(t')dt'\Big) \quad (4)$$

where $x_0 = x(t_0)$.

Example

Ex.: $x' = \sin(t)x \quad (a(t) = \sin t)$

$$\Rightarrow \int a(t)\,dt = \int \sin(t)\,dt = -\cos t$$

$$\Rightarrow x(t) = Ce^{-\cos t}$$

Example

Ex.: $x' = x/t \quad (a(t) = 1/t)$

$$\Rightarrow \int a(t)\,dt = \int \frac{dt}{t} = \ln|t|$$

$$\Rightarrow \quad x(t) = Ce^{\ln|t|} = C|t| \quad (t \neq 0)$$

Since either $t > 0$ or $t < 0$:

$$\Rightarrow \quad x(t) = Bt \quad (t \neq 0,\ B = \pm C)$$

Nonhomogeneous Equations: Integrating Factor

NHE: $x' - a(t)x = f(t)$ (5)

Multiply by **integrating factor** $u(t)$ (determined below):

$$u(t)x' - u(t)a(t)x = u(t)f(t) \quad (6)$$

If $u(t)$ satisfies

$$u' = -a(t)u \quad (7)$$

then

$$ux' - uax = ux' + u'x = (ux)'$$

$$\Rightarrow \quad (u(t)x)' = u(t)f(t)$$

$$\Rightarrow u(t)x(t) = \int u(t)f(t)dt + C$$

$\Rightarrow$ **General Solution:**

$$x(t) = \frac{1}{u(t)} \int u(t)f(t)dt + C/u(t) \quad (8)$$

where $u(t)$ is a (part.) solution to the HE (7) (cf. (2) and (3)):

$$u(t) = \exp\Big(-\int a(t)dt\Big) \quad (9)$$

Example

Ex.: $x' - x = e^{-t}$ $(a(t) = 1,\ f(t) = e^{-t})$

$$\int a(t)\,dt = \int dt = t \Rightarrow u(t) = e^{-t}$$

$$ux' - ux = e^{-t}x' - e^{-t}x = (e^{-t}x)',\ ue^{-t} = e^{-2t}$$

$$(e^{-t}x)' = e^{-2t} \Rightarrow e^{-t}x = \int e^{-2t}dt$$

$$e^{-t}x = -e^{-2t}/2 + C \Rightarrow x(t) = -e^{-t}/2 + Ce^{t}$$

Nonhomogeneous Equations: Variation of Parameter

Set
$$x_h(t) = 1/u(t) \quad (10)$$
$$= \exp\left(\int a(t)dt\right)$$

and rewrite (8) as

$$x(t) = Cx_h(t) + x_p(t) \quad (11)$$

where

$$x_p(t) = x_h(t)v(t) \quad (12)$$

with

$$v(t) = \int \frac{f(t)}{x_h(t)}\,dt \quad (13)$$

Eq (12) with (13) is called **Variation of Parameter Formula**

Terms in Gen. Sol. (11):

$Cx_h(t)$: Gen. Sol. of HE (2)

$x_p(t)$: Part. Sol. of NHE (5)

Example

Ex.: $x' = x \tan t + \sin t$

HE: $x' = x \tan t \ \Rightarrow \ x_h(t) = \exp\left(\int \tan t \, dt\right)$

$\Rightarrow \ x_h(t) = \exp(\ln(1/\cos t)) = 1/\cos t$

$$\begin{aligned} v(t) &= \int [f(t)/x_h(t)]\, dt \\ &= \int \sin t \cos t \, dt = -\cos^2 t/2 \end{aligned}$$

$$\begin{aligned} x_p(t) &= x_h(t) v(t) = (1/\cos t)(-\cos^2 t/2) \\ &= -\cos t/2 \end{aligned}$$

$\Rightarrow$ Gen. Sol.: $x(t) = -\cos t/2 + C/\cos t$

Example

Example: $y' - ry = f(t)$; use variation of parameter

$$y_h(t) = \exp(\int r\,dt) = e^{rt}, \quad v(t) = \int [f(t)/y_h(t)]dt = \int e^{-rt} f(t)\,dt$$

$$\Rightarrow \text{ gen. sol.: } y(t) = e^{rt}(\int e^{-rt} f(t)\,dt + C)$$

If $f(t) = a = \text{const} \Rightarrow \int e^{-rt} f(t)\,dt = a \int e^{-rt} dt = -(a/r)e^{-rt} \Rightarrow y(t) = Ce^{rt} - a/r$

Exercise 2.4.3

Ex. 3: $y' + (2/x)y = (\cos x)/x^2$; use integrating factor

$$a(x) = -2/x \;\Rightarrow\; u(x) = \exp(-\int a(x)dx) = \exp(2\ln x) = \exp(\ln x^2) = x^2$$

Multiply ODE by $x^2 \Rightarrow x^2y' + 2xy = \cos x \;\Rightarrow\; (x^2y)' = \cos x \;\Rightarrow\; x^2y = \int \cos x\, dx$

$$\Rightarrow\; x^2y = \sin x + C \;\Rightarrow\; y(x) = (\sin x + C)/x^2$$

Exercise 2.4.5

Ex. 5: $x' - 2x/(t+1) = (t+1)^2$; use variation of parameter

HE: $x' = 2x/(t+1) \;\Rightarrow\; x_h(t) = \exp(\int [2/(t+1)]dt) = \exp(\ln[(t+1)^2]) = (t+1)^2$

$v(t) = \int [f(t)/x_h(t)]dt = \int dt = t \;\Rightarrow$ part. sol.: $x_p(t) = x_h(t)v(t) = (t+1)^2 t$

gen. sol.: $x(t) = x_p(t) + Cx_h(t) = (t+1)^2(t+C)$

Exercise 2.4.7

Ex. 7: $(1+x)y' + y = \cos x \Rightarrow y' + y/(1+x) = (\cos x)/(1+x)$; use int. factor

$$u(x) = \exp(-\int a(x)dx) = \exp(\int [1/(1+x)]\, dx) = 1 + x$$

$$\text{Multiply ODE by } u(x) \;\Rightarrow\; (1+x)y' + y = [(1+x)y]' = \cos x$$

$$\Rightarrow\; (1+x)y = \int \cos x\, dx = \sin x + C \;\Rightarrow\; y(x) = (\sin x + C)/(1+x)$$

Exercise 2.4.11

Ex. 11: $y' = \cos x - y\sec x$; use variation of parameter

$$\text{HE: } y' = -y\sec x \;\Rightarrow\; y_h(x) = \exp(-\int \sec x\,dx)$$

$$\Rightarrow\; y_h(x) = \exp(-\ln(\sec x + \tan x)) = 1/(\sec x + \tan x)$$

$$\Rightarrow\; v(x) = \int [f(x)/y_h(x)]dx = \int (\sec x + \tan x)\cos x\,dx = \int (1+\sin x)dx = x - \cos x$$

$$\Rightarrow\; y_p(x) = y_h(x)v(x) = (x - \cos x)/(\sec x + \tan x)$$

$$\Rightarrow\; y(x) = y_p(x) + Cy_h(x) = (x - \cos x + C)/(\sec x + \tan x)$$

Exercise 2.4.12

Ex. 12: $x' - (n/t)x = e^t t^n$; use integrating factor

$$u(t) = \exp(-\int (n/t)dt) = \exp(-n \ln t) = \exp(\ln t^{-n}) = t^{-n}$$

Multiply ODE by $u(t) \Rightarrow (t^{-n}x)' = e^t \Rightarrow (t^{-n}x) = \int e^t dt = e^t + C$

$$\Rightarrow \quad x(t) = t^n(e^t + C)$$

Exercise 2.4.33

Ex. 33: $ty' + y = 4t^2$

Here one sees directly: $(ty)' = ty' + y = 4t^2 \Rightarrow ty = (4/3)t^3 + C$

$$\Rightarrow \quad y(t) = (4/3)t^2 + C/t$$

Exercise 2.4.35

Ex. 35: $y' + 2xy = 4x$; use variation of parameter

$$y_h(x) = \exp\left(\int (-2x)dx\right) = e^{-x^2} \Rightarrow v(x) = \int 4xe^{x^2}dx = 2e^{x^2}$$

$$\Rightarrow \quad y_p(x) = y_h(x)v(x) = 2 \Rightarrow y(x) = 2 + Ce^{-x^2}$$

Exercise 2.4.15

Ex. 15: $(x^2+1)y' + 3xy = 6x$, $y(0) = -1$; use integrating factor

normal form: $y' + [3x/(1+x^2)]y = 6x/(1+x^2) \;\Rightarrow\; u(x) = \exp(\int [3x/(1+x^2)]dx)$

$$\Rightarrow\; u(x) = \exp((3/2)\ln(1+x^2)) = (1+x^2)^{3/2} \;\Rightarrow\; (uy)' = 6x(1+x^2)^{1/2}$$

$$\Rightarrow\; uy = \int 6x(1+x^2)^{1/2}dx = 2(1+x^2)^{3/2} + C \;\Rightarrow\; y(x) = 2 + C(1+x^2)^{-3/2}$$

Invoke IC: $y(0) = 2 + C = -1 \;\Rightarrow\; C = -3 \;\Rightarrow\; y(x) = 2 - 3(1+x^2)^{-3/2}$

Exercise 2.4.17

Ex. 17: $x' + x\cos t = (1/2)\sin 2t = \sin t\cos t,\ x(0) = 1$; use variation of parameter

$$x_h(t) = \exp(-\int \cos t\,dt) = e^{-\sin t} \Rightarrow v(t) = \int \sin t\cos t e^{\sin t} dt = e^{\sin t}(\sin t - 1)$$

$$\Rightarrow x_p(t) = \sin t - 1 \Rightarrow \text{gen. sol.: } x(t) = \sin t - 1 + Ce^{-\sin t}$$

Invoke IC: $x(0) = -1 + C = 1 \Rightarrow C = 2 \Rightarrow x(t) = \sin t - 1 + 2e^{-\sin t}$

Exercise 2.4.19

Ex. 19: $(2x+3)y' = y + (2x+3)^{1/2}$, $y(-1) = 0$; use integrating factor

normal form: $y' = y/(2x+3) + (2x+3)^{-1/2} \Rightarrow u(x) = \exp(\int [-1/(2x+3)]dx)$

$$\Rightarrow u(x) = \exp(-(1/2)\ln(2x+3)) = (2x+3)^{-1/2} \Rightarrow (uy)' = (2x+3)^{-1}$$

$$\Rightarrow uy = \int (2x+3)^{-1} dx = (1/2)\ln(2x+3) + C$$

$$\Rightarrow y(x) = (1/2)(2x+3)^{1/2}(\ln(2x+3) + C), \text{ IC } \Rightarrow y(-1) = C/2 = 0$$

$$\Rightarrow y(x) = (1/2)(2x+3)^{1/2}\ln(2x+3), \text{ IoE: } (-3/2, \infty)$$

Exercise 2.4.21

Ex. 21: $(1+t)x' + x = \cos t,\ x(-\pi/2) = 0$

One sees directly: $[(1+t)x]' = (1+t)x' + x = \cos t \Rightarrow (1+t)x = \int \cos t\, dt$

$$\Rightarrow (1+t)x = \sin t + C \ \Rightarrow\ x(t) = (\sin t + C)/(1+t)$$

$x(-\pi/2) = (-1+C)/(1-\pi/2) = 0 \ \Rightarrow\ C = 1 \ \Rightarrow\ x(t) = (1+\sin t)/(1+t),$ IoE: $(-\infty, -1)$

Exercise 2.4.37

Ex. 37: $y' + y/2 = t,\ y(0) = 1;$

From Example p.7 ($r = -1/2,\ f(t) = t$): $y(t) = e^{-t/2}(\int e^{t/2} t\,dt + C)$

$$\int e^{t/2} t\,dt = 2(t-2)e^{t/2} \Rightarrow y(t) = 2(t-2) + Ce^{-t/2}$$

IC: $y(0) = -4 + C = 1 \Rightarrow C = 5 \Rightarrow y(t) = 2(t-2) + 5e^{-t/2}$

Exercise 2.4.39

Ex. 39: $y' + 2xy = 2x^3$, $y(0) = -1$; use variation of parameter

$$y_h(x) = \exp\left(\int(-2x)dx\right) = e^{-x^2} \Rightarrow v(x) = \int 2x^3 e^{x^2} dx = (x^2-1)e^{x^2} \Rightarrow y_p(x) = x^2-1$$

$$\Rightarrow y(x) = x^2 - 1 + Ce^{-x^2}, \text{ IC: } y(0) = C - 1 = -1 \Rightarrow C = 0 \Rightarrow y(x) = x^2 - 1$$