Math 3331 Differential Equations

8.2 Geometric Interpretation of Solutions

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8.2 Geometric Interpretation of Solutions

- Definitions
  - Autonomous system and Phase Space Plot
  - Planar Autonomous System
- Example: Predator-Prey System
- Worked out Examples from Exercises:
  - 1, 3, 17, 22, 23, 24, 25

Autonomous system and Phase Space Plot

Autonomous system:

$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} = [x_1, \ldots, x_n]^T$

For any t: $\mathbf{x}(t) \in \mathbf{R}^n$

$\Rightarrow$ RHS doesn't depend explicitly on t.

- $\mathbf{R}^n$: **phase space** ($n = 2$: phase plane)
- **Trajectory:** Curve

  $\{\mathbf{x}(t) \,|\, t \in I\}$ in $\mathbf{R}^n$

  I: interval on which $\mathbf{x}(t)$ is defined

- **Tangent vectors:**

  $$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$$

- **Vector field:** $\mathbf{x} \to \mathbf{f}(\mathbf{x})$

- $\mathbf{x}(t)$ solution $\Rightarrow$ $\mathbf{x}(t - t_0)$ solution: *same trajectory!*
- If existence and uniqueness, trajectories don't intersect

Planar Autonomous System

Planar Autonomous system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

$\Rightarrow$ RHSs f and g don't depend explicitly on t.

- The xy-plane is the **phase plane**
- The solution curve $t \to (x(t), y(t))$ is a **trajectory** (or **phase plane plot**).

- Tangent vector:

$$(x'(t), y'(t)) = (f(x(t), y(t)), g(x(t), y(t)))$$

- Vector field:

$$(x, y) \to (f(x, y), g(x, y))$$

Predator-Prey System

Example: Lotka-Volterra's predator-prey equations

$$\begin{aligned} R' &= (a - bF)R \\ F' &= (-c + dR)F \end{aligned} \qquad (1)$$

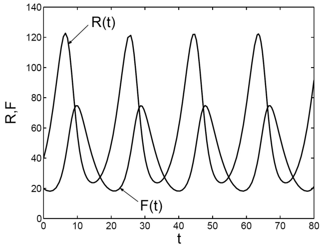
$$a, b, c, d > 0$$

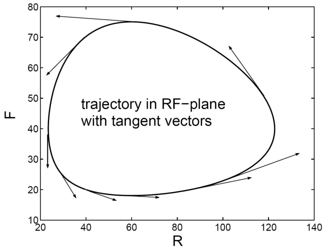
R: number of rabbits
F: number of foxes

Parameters:

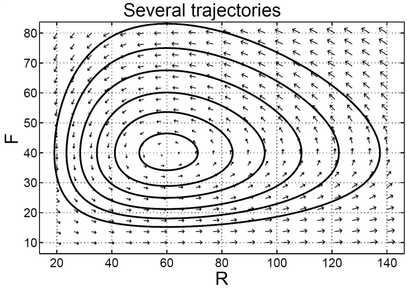
$$\begin{aligned} a = 0.4,\ b = 0.01 \\ c = 0.3,\ d = 0.005 \end{aligned} \qquad (2)$$

IC: $R(0) = 40,\ F(0) = 20$

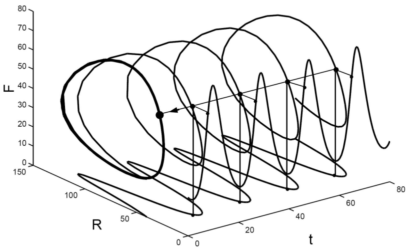




Predator-Prey System (cont.)



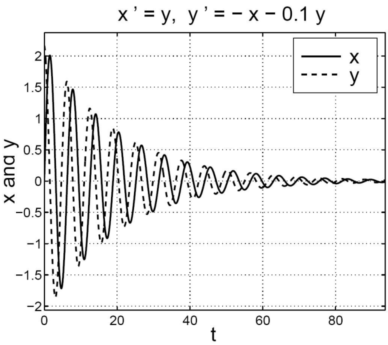
Composite Graph



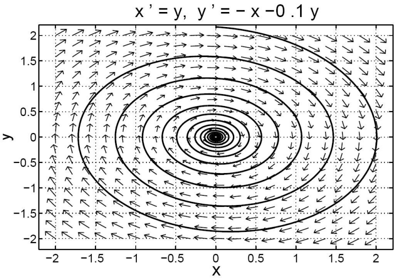
Example

Ex.:
$$\begin{aligned} x' &= y \\ y' &= -x - 0.1y \end{aligned}$$
$$x(0) = 0,\ y(0) = 2$$

time plots



trajectory and vector field



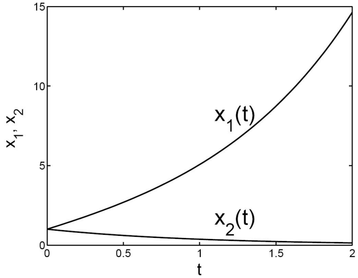
Exercise 8.2.1

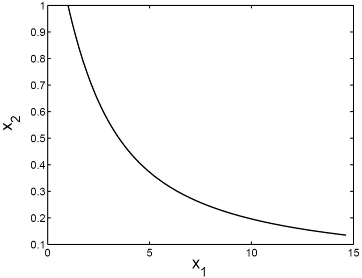
Ex. 8.2.1: Plot (i) $x_1(t)$,$x_2(t)$ and (ii) the curve $t \to (x_1(t), x_2(t))$ for

$$\mathbf{x}(t) = [2e^t - e^{-t}, e^{-t}]^T, \text{ i.e. } x_1(t) = 2e^t - e^{-t},\ x_2(t) = e^{-t}$$

Matlab commands:

```
t=linspace(0,2,100);x1=2*exp(t)-exp(-t);x2=exp(-t);
figure(1),plot(t,x1,'k',t,x2,'k--'),xlabel('t'),ylabel('x_1 and x_2')
figure(2),plot(x1,x2,'k'),xlabel('x_1'),ylabel('x_2'),axis([0 15 0 1])
```

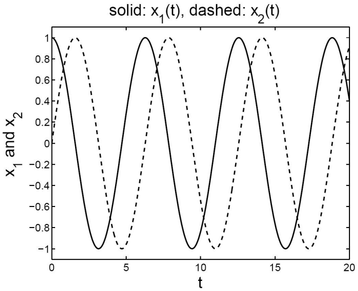


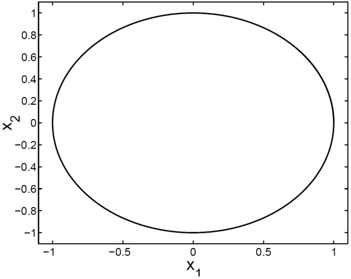


Exercise 8.2.3

Ex. 8.2.3: Same as Ex. 8.2.1 for

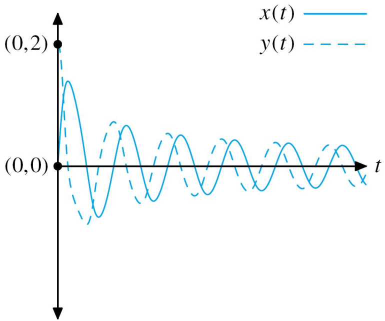
$$\mathbf{x}(t) = [\cos t, \sin t]^T, \text{ i.e. } x_1(t) = \cos t,\ x_2(t) = \sin t$$



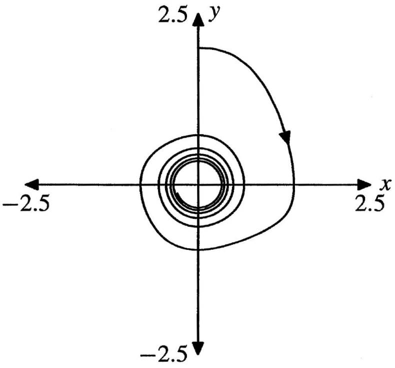


Exercise 8.2.22

Plot $t \to (x(t), y(t))$ in the xy-plane.

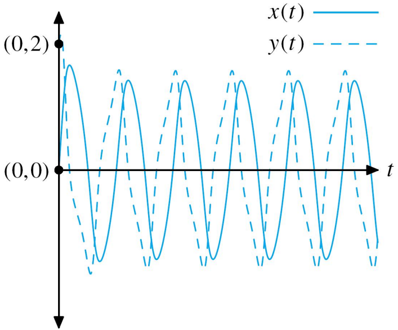


Initially, $x(0) = 0$ and $y(0) = 2$, then y decays as x increases, thereafter both x and y oscillate as they decay toward zero.

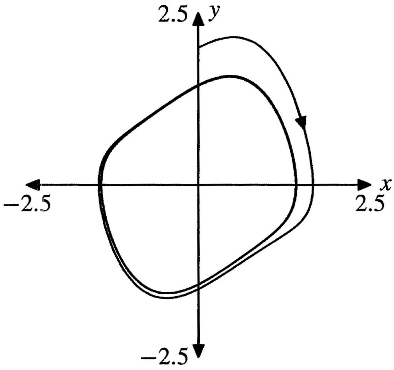


Exercise 8.2.23

Plot $t \to (x(t), y(t))$ in the xy-plane.

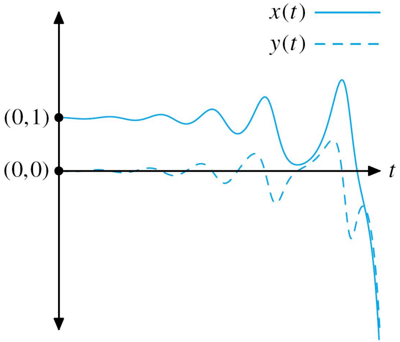


Initially, $x(0) = 0$ and $y(0) = 2$. Shortly thereafter, y decays as x increases. Soon, both x and y begin a seemingly periodic motion.

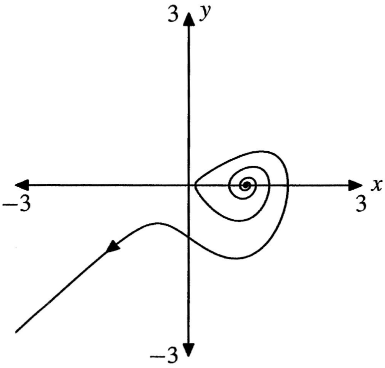


Exercise 8.2.24

Plot $t \to (x(t), y(t))$ in the xy-plane.

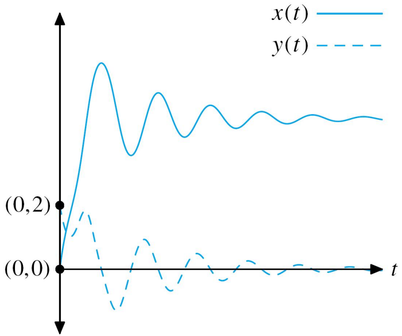


At first, x oscillates mildly about 1, while y oscillates mildly about zero. This would indicate a turning about $(1, 0)$ in the phase plane. The oscillations grow larger until both x and y shoot off to $-\infty$. One possible solution follows.

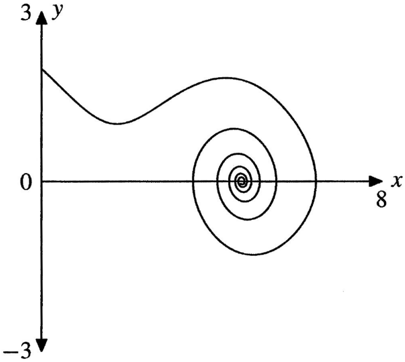


Exercise 8.2.25

Plot $t \to (x(t), y(t))$ in the xy-plane.



Initially, $x(0) = 0$ and $y(0) = 2$. Thereafter, x increases rapidly, then decays asymptotically in an oscillatory manner to about 5 or 6. Meanwhile, y decays, eventually oscillating about zero. One possible solution follows.



Exercise 8.2.17

Ex. 8.2.17: Plot (i) solutions $x(t), y(t)$ of IVP as functions of t, (ii) trajectory

IVP: $x' = -6x + 10y,\ y' = -5x + 4y,\ x(0) = 5,\ y(0) = 1.$ Use *pplane6*: