Math 3331 Differential Equations

8.3 Qualitative Analysis

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8.3 Qualitative Analysis

- Equilibrium Points
  - Equilibrium Points and Nullclines
  - Examples
- Worked out Examples from Exercises:
  - 1, 2, 7

Equilibrium Points and Nullclines

$$\textbf{Ex.}: \quad \begin{aligned} R' &= (a - bF)R \\ F' &= (-c + dR)F \end{aligned}$$

Equilibrium points: $R' = F' = 0$

$$\Rightarrow \begin{cases} (a - bF)R &= 0 \\ (-c + dR)F &= 0 \end{cases}$$

Solutions:

$[R, F]^T = [0, 0]^T, \ \ [R, F]^T = [c/d, a/b]^T$

Equilibrium points $\rightarrow$
constant solutions of ODE-system:

$$[R(t), F(t)]^T = [c/d, a/b]^T$$

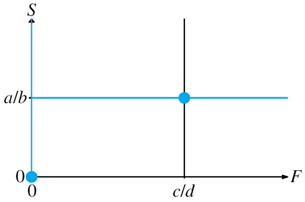
R-nullcline: $R' = 0$

$$\Rightarrow \quad R = 0 \text{ and } F = a/b$$

F-nullcline: $F' = 0$

$$\Rightarrow \quad F = 0 \text{ and } R = c/d$$

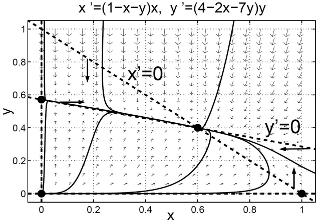
Equilibrium points are intersections of nullclines



Example

Ex.:
$$\begin{aligned} x' &= (1-x-y)x \\ y' &= (4-2x-7y)y \end{aligned}$$

several solutions, nullclines, and equilibrium points (using *pplane6*)



x-nullclines: $x = 0$, $x + y = 1$

y-nullclines: $y = 0$, $2x + 7y = 4$

Equlibrium points:

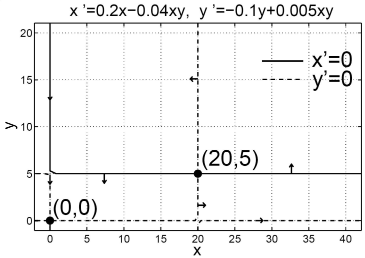
$(0,0)$, $(0,4/7)$, $(1,0)$, $(3/5,2/5)$

Exercise 8.3.1

Ex. 8.3.1: Plot (i) nullclines and (ii) equilibrium points for

$$\left\{ \begin{array}{rcl} x' & = & 0.2x - 0.04xy \\ y' & = & -0.1y + 0.005xy \end{array} \right\}. \text{ Nullclines: } \left\{ \begin{array}{lcl} x' = 0 & \Rightarrow & x = 0 \text{ and } y = 5 \\ y' = 0 & \Rightarrow & y = 0 \text{ and } x = 20 \end{array} \right\}$$

Equilibria: $\left\{ \begin{array}{c} [0,0]^T \\ [20,5]^T \end{array} \right\}$ Use *pplane6*:

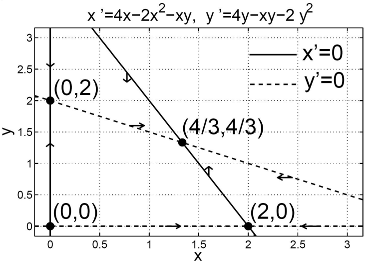


Exercise 8.3.2

Ex. 8.3.2: Plot (i) nullclines and (ii) equilibrium points for

$$\left\{ \begin{array}{lcl} x' & = & 4x - 2x^2 - xy \\ y' & = & 4y - xy - 2y^2 \end{array} \right\}. \text{ Nullclines: } \left\{ \begin{array}{lcl} x' = 0 & \Rightarrow & x = 0 \text{ and } 2x + y = 4 \\ y' = 0 & \Rightarrow & y = 0 \text{ and } x + 2y = 4 \end{array} \right\}$$

Equilibria: $\left\{ \begin{array}{cc} [0,0]^T & [2,0]^T \\ [4/3,4/3]^T & [0,2]^T \end{array} \right\}$. *pplane6*:



Exercise 8.3.7(a)

Ex. 8.3.7: Consider $\left\{ \begin{array}{lcl} x' & = & 1 - (y - \sin x)\cos x \\ y' & = & \cos x - y + \sin x \end{array} \right\}$

(a) Show that $x(t) = t$, $y(t) = \sin t$ is solution:

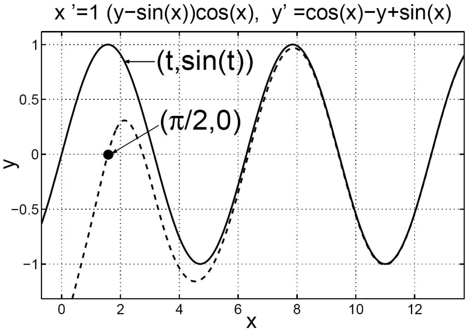
$x' = 1$, $\left\{ \begin{array}{c} 1 - (y - \sin x)\cos x = \\ 1 - (\sin t - \sin t)\cos t = 1 \end{array} \right\}$ OK

$y' = \cos t$, $\left\{ \begin{array}{c} \cos x - y + \sin x = \\ \cos t - \sin t + \sin t = \cos t \end{array} \right\}$ OK

Exercise 8.3.7(b)

Ex. 8.3.7: Consider $\left\{ \begin{array}{rcl} x' & = & 1 - (y - \sin x)\cos x \\ y' & = & \cos x - y + \sin x \end{array} \right\}$

(b) Plot solutions:



Exercise 8.3.7(c)

Ex. 8.3.7: Consider $\left\{ \begin{array}{lcl} x' & = & 1-(y-\sin x)\cos x \\ y' & = & \cos x - y + \sin x \end{array} \right\}$

(c) Show that $y(t) < \sin x(t)$ for all t if $x(0) = \pi/2$, $y(0) = 0$:

Solution of (a) satisfies $y = \sin x$. Trajectories don't cross $\Rightarrow y(t) < \sin x(t)$ if $y(0) < \sin x(0)$.