# Math 3331 Differential Equations

## 8.5 Properties of Linear Systems

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# 8.5 Properties of Linear Systems

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## Superposition Principle

$$\mathbf{x}' = A(t)\mathbf{x} \qquad (3)$$

$$A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y} \quad \Rightarrow$$

**Thm.:** (Superposition Principle)
If $\mathbf{x}_1(t), \mathbf{x}_2(t)$ are solutions of (3) and $c_1, c_2$ are arbitrary constants, then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

is also a solution.

# Superposition Principle (cont.)

Superposition principle does in general **not** hold for

- nonlinear systems
- nonhomogeneous linear systems

# Example 1

**Ex.:** $x' = x^2 \rightarrow$ solution $x(t) = -1/t$.
$y(t) = -x(t) = 1/t$ is *not* solution,
because $y' = -1/t^2$, $y^2 = 1/t^2$.

# Example 2

**Ex.:** $x' = x - 1 \rightarrow$ solution $x(t) = 1$.
$y(t) = 0 \cdot x(t) = 0$ is *not* solution.

# Example 3

**Ex.:** $\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \ \mathbf{x}_2(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

are solutions (verify by substitution)

$$\Rightarrow \mathbf{x}(t) = c_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

is solution for any $c_1, c_2$. Rewrite:

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix} \mathbf{c}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

# Example 3 (cont.)

Consider IC: $\mathbf{x}(0) = \mathbf{x}_0 \Rightarrow$

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{c} = \mathbf{x}_0$$

Invert matrix: $\mathbf{c} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}_0$

$\Rightarrow$ unique solution for *any* $\mathbf{x}_0$

e.g.: $\mathbf{x}_0 = [3, -1]^T \Rightarrow \mathbf{c} = [2, 1]^T \Rightarrow$

$$\mathbf{x}(t) = 2\mathbf{x}_1(t) + \mathbf{x}_2(t) = \begin{bmatrix} 2e^{-t} + e^{3t} \\ -2e^{-t} + e^{3t} \end{bmatrix}$$

# Linear Independence

**Thm.:** Assume $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are $k$ solutions of

$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x} \in \mathbf{R}^n \tag{3}$$

for $t$ on $I$ and that $a_{ij}(t)$ are continuous on $I$. Let $t_0 \in I$.

1. If there are constants $c_1, \ldots, c_k$, not all 0, such that $c_1\mathbf{x}_1(t_0) + \cdots + c_k\mathbf{x}_k(t_0) = \mathbf{0}$, then $c_1\mathbf{x}_1(t) + \cdots + c_k\mathbf{x}_k(t) \equiv \mathbf{0}$.
2. If the vectors $\mathbf{x}_1(t_0), \ldots, \mathbf{x}_k(t_0)$ are linearly independent, then $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are linearly independent for any $t$ on $I$.

# Fundamental Set of Solutions

**Def.:** (Fundamental Set) Assume $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are $n$ solutions of (3) on an interval $I$ on which $A(t)$ is continuous. $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are called a fundamental set of solutions if the vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are linearly independent for all $t$ on $I$.

**Note:** Thm. $\Rightarrow$ it is sufficient that $\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0)$ are linearly independent for some $t_0$.

# Fundamental Matrix and Wronskian

- Find fundamental set $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ (Ch. 9)
- General solution: $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$
- Rewrite this as

$$\mathbf{x}(t) = X(t)\mathbf{c}$$

$$\mathbf{c} = [c_1, \ldots, c_n]^T$$

$$X(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)]$$

- $X(t)$ $(n \times n)$ is called **fundamental matrix**
- Match $\mathbf{c}$ to IC:

$$\mathbf{x}(t_0) = X(t_0)\mathbf{c} = \mathbf{x}_0$$

$$\Rightarrow \mathbf{c} = \left(X(t_0)\right)^{-1}\mathbf{x}_0$$

**Wronskian:** $W(t) = \det(X(t))$

Condition for linear independence: $W(t_0) \neq 0$

# Nonhomogeneous System: Solution Strategy

**Nonhomogeneous System:**

Given a particular solution $\mathbf{x}_p(t)$, any solution $\mathbf{x}(t)$ of

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$$

can be written in the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + X(t)\mathbf{c}$$

# Example 1

**Ex.:** $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$. Solutions:

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \ \mathbf{x}_2(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

$$\Rightarrow X(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix}$$

Wronskian of $\mathbf{x}_1(t), \mathbf{x}_2(t)$:

$$\begin{aligned} W(t) &= \det(X(t)) \\ &= e^{-t}e^{3t} + e^{-t}e^{3t} = 2e^{2t} \neq 0 \end{aligned}$$

$\Rightarrow X(t)$ is fundamental matrix.

# Exercise 8.5.4

**Ex. 8.5.4:** Rewrite system using matrix notation

$$\left\{ \begin{array}{ccc} x_1' & = & -x_2 \\ x_2' & = & x_1 \end{array} \right\} \to \mathbf{x}' = A\mathbf{x} \text{ with } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

# Exercise 8.5.6

**Ex. 8.5.6:** Rewrite system using matrix notation

$$\left\{ \begin{array}{rcl} x_1' & = & -x_2 + \sin t \\ x_2' & = & x_1 \end{array} \right\} \to \mathbf{x}' = A\mathbf{x} + \mathbf{f}(t) \text{ with } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; \mathbf{f}(t) = \begin{bmatrix} \sin t \\ 0 \end{bmatrix}$$

# Exercise 8.5.10

**Ex. 8.5.10:** Let $\mathbf{x}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, $\mathbf{y}(t) = \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$

Show that $\mathbf{x}(t), \mathbf{y}(t)$ are solutions of the system of Ex. 8.5.4. Verify that any linear combination is a solution.

1. $\mathbf{x}(t) \to x_1(t) = \cos t,\ x_2(t) = \sin t.\ \ x_1' = -\sin t = -x_2,\ x_2' = \cos t = x_1$: OK.
2. $\mathbf{y}(t) \to y_1(t) = \sin t,\ y_2(t) = -\cos t.\ \ y_1' = \cos t = -y_2,\ y_2' = \sin t = y_1$: OK.
3. $(c_1\mathbf{x}(t)+c_2\mathbf{y}(t))' = c_1\mathbf{x}'(t)+c_2\mathbf{y}'(t) = c_1 A\mathbf{x}(t)+c_2 A\mathbf{y}(t) = A(c_1\mathbf{x}(t)+c_2\mathbf{y}(t))$

# Exercise 8.5.12

**Ex. 8.5.12:** Let $\mathbf{x}_p(t) = \frac{1}{2}\begin{bmatrix} t\sin t - \cos t \\ -t\cos t \end{bmatrix}$

Show that $\mathbf{x}_p(t)$ is a solution of the system of Ex. 8.5.6. Further show that $\mathbf{z}(t) = \mathbf{x}_p(t) + c_1\mathbf{x}(t) + c_2\mathbf{y}(t)$ is also solution, where $\mathbf{x}(t), \mathbf{y}(t)$ are from Ex. 8.5.10.

1. $\mathbf{x}_p(t) \to x_1(t) = (t\sin t - \cos t)/2$, $x_2(t) = -(t\cos t)/2$.
   - a: $x_1'(t) = (t\cos t + \sin t)/2 + (\sin t)/2 = (t\cos t)/2 + \sin t$
     $-x_2(t) + \sin t = (t\cos t)/2 + \sin t$: OK
   - b: $x_2'(t) = -(\cos t)/2 + (t\sin t)/2 = x_1(t)$: OK
2. $\mathbf{z}'(t) = \mathbf{x}_p'(t) + c_1\mathbf{x}'(t) + c_2\mathbf{y}'(t) = (A\mathbf{x}_p(t) + \mathbf{f}(t)) + c_1 A\mathbf{x}(t) + c_2 A\mathbf{y}(t)$
   $= A(\mathbf{x}_p(t) + c_1\mathbf{x}(t) + c_2\mathbf{y}(t)) + \mathbf{f}(t)) = A\mathbf{z}(t) + \mathbf{f}(t)$

# Exercise 8.5.18

**Ex. 8.5.18:** Let $\mathbf{y}_1(t) = \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$, $\mathbf{y}_2(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$

Suppose that $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$ are solutions of a homogeneous linear system. Further suppose that $\mathbf{x}(t)$ is a solution of the same system with IC $\mathbf{x}(0) = [1, -1]^T$. Find $c_1, c_2$ such that $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

$$\text{Let } Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] \Rightarrow Y(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{c} = (Y(0))^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \Rightarrow c_1 = 2,\ c_2 = -3$$

# Exercise 8.5.19

**Ex. 8.5.19:** Let $\mathbf{y}_1(t) = \begin{bmatrix} -e^{-t} \\ -e^{-t} \\ e^{-t} \end{bmatrix}$, $\mathbf{y}_2(t) = \begin{bmatrix} 0 \\ -e^{t} \\ 2e^{t} \end{bmatrix}$, $\mathbf{y}_3(t) = \begin{bmatrix} e^{2t} \\ 0 \\ 2e^{2t} \end{bmatrix}$

$\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t)$ are solutions of a homogeneous linear system. Check linear dependence or independence of these solutions.

Let $Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t)]$. It is sufficient to check for $t = 0$. Wronskian:

$$\begin{aligned} W(0) &= \det(Y(0)) = \begin{vmatrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} \\ &= (-1)^{2+1}(-1)\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} + (-1)^{2+2}(-1)\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -2 + 3 = 1 \end{aligned}$$

$\Rightarrow \mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t)$ are linearly independent for all $t$.

# Exercise 8.5.19 (cont.)

Confirm this using Matlab's symbolic toolbox:

```
syms t;y1=[-exp(-t);-exp(-t);exp(-t)];y2=[0;-exp(t);2*exp(t)];
y3=[exp(2*t);0;2*exp(2*t)];Y=[y1 y2 y3];simplify(det(Y))
```

Answer in Command Window:

```
>> exp(2*t)
```

Hence $W(t) = e^{2t}$ which is indeed nonzero for all $t$.