Math 3331 Differential Equations

9.6 The Exponential of a Matrix

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math3331

9.6 The Exponential of a Matrix

- Fundamental Matrix
- Matrix Exponential
- Properties of the Matrix Exponential
- Matrices With Only One Eigenvalue
- Generalized Eigenvectors: Definition
- Generalized Eigenvectors and Associated Solutions
- Examples

Fundamental Matrix

$$\mathbf{x}' = A\mathbf{x}, \quad A : n \times n \qquad (1)$$

Def.: If $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ is a fundamental set of solutions (F.S.S.) of (1), then

$$X(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)] \ (n \times n)$$

is called a fundamental matrix (F.M.) for (1).

General solution: $(\mathbf{c} = [c_1, \ldots, c_n]^T)$

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) \\ &= X(t)\mathbf{c} \end{aligned}$$

Thm.: If $X(t)$ is a F.M. for (1) and C is a constant nonsingular matrix, then $X(t)C$ is also a F.M.

Proof: Each column of $X(t)C$ is a linear combination of the columns of $X(t)$ and so is a solution of (1), and $X(0)C$ is nonsingular.

Example

Ex.: $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1 \quad \leftrightarrow \quad \mathbf{v}_1 = [2, 3]^T$$
$$\lambda_2 = -2 \quad \leftrightarrow \quad \mathbf{v}_2 = [1, 1]^T$$

F.S.S.:

$$\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{x}_2(t) = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

F.M.: $X(t) = \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix}$

If we set

$$\mathbf{y}_1(t) = 2\mathbf{x}_2(t), \quad \mathbf{y}_2(t) = 3\mathbf{x}_2(t),$$

$\mathbf{y}_1(t), \mathbf{y}_2(t)$ are also F.S.S. with F.M.

$$Y(t) = \begin{bmatrix} 3e^{-2t} & 4e^{-t} \\ 3e^{-2t} & 6e^{-t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$$

Matrix Exponential

Consider IVP:

$$\mathbf{x}' = A\mathbf{x}, \ \mathbf{x}(0) = \mathbf{x}_0 \qquad (2)$$

Solution of IVP: If $X(t)$ is a F.M., the general solution is

$$\mathbf{x}(t) = X(t)\mathbf{c}$$

Match $\mathbf{c}$ to IC:

$$\begin{aligned} \mathbf{x}(0) &= X(0)\mathbf{c} = \mathbf{x}_0 \\ \Rightarrow \quad \mathbf{c} &= (X(0))^{-1}\mathbf{x}_0 \\ \Rightarrow \quad \mathbf{x}(t) &= X(t)(X(0))^{-1}\mathbf{x}_0 \end{aligned}$$

Def.: Given a F.M. $X(t)$, then

$$e^{At} \overset{def}{=} X(t)(X(0))^{-1}$$

is the matrix exponential of At.

Thm.: The solution of (2) is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

Example: Matrix Exponential

Ex.: $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$

F.M.: $X(t) = \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix}$

$X(0) = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$, $(X(0))^{-1} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

$$\begin{aligned} e^{At} &= X(t)(X(0))^{-1} \\ &= \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-2t} - 2e^{-t} & 2e^{-t} - 2e^{-2t} \\ 3e^{-2t} - 3e^{-t} & 3e^{-t} - 2e^{-2t} \end{bmatrix} \end{aligned}$$

Example: IVP

$$\textbf{IVP:} \quad \mathbf{x}' = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \mathbf{x}, \; \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solution:

$$\mathbf{x}(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-t} & 2e^{-t} - 2e^{-2t} \\ 3e^{-2t} - 3e^{-t} & 3e^{-t} - 2e^{-2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{-2t} - 2e^{-t} \\ 4e^{-2t} - 3e^{-t} \end{bmatrix}$$

Properties of the Matrix Exponential (I)

- Exponential series $(A^0 = I)$:

$$e^{At} = \sum_{m=0}^{\infty} (At)^m/m!$$

Convergence for any matrix A

- $$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

- If $D = [d_{ij}]$ is a diagonal matrix ($d_{ij} = 0$ for $i \neq j$), then e^{Dt} is a diagonal matrix with entries $e^{d_{ii}t}$. Ex.:

$$\exp\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} t\right) = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

- Special case ($d_{ii} = r$):

$$e^{(rI)t} = e^{rt}I$$

Properties of the Matrix Exponential (II)

- If $AB = BA$, then

$$e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At}$$

Note: If $AB \neq BA$, then in general

$$e^{(A+B)t} \neq e^{At}e^{Bt} \neq e^{Bt}e^{At}$$

- e^{At} is nonsingular, and

$$(e^{At})^{-1} = e^{-At}$$

- If V is nonsingular, then

$$e^{(VAV^{-1})t} = Ve^{At}V^{-1}$$

- If $\mathbf{v}$ is an eigenvector for an eigenvalue λ, then

$$e^{At}\mathbf{v} = e^{\lambda t}\mathbf{v}$$

Matrices With Only One Eigenvalue

Thm.: If A has only one eigenvalue λ, then there is an integer k, $0 < k \le n$, such that

$$(A - \lambda I)^k = 0$$

Use this to compute e^{At} as follows. Write $A = \lambda I + (A - \lambda I)$. Then

$$\begin{aligned} e^{At} &= e^{(\lambda I)t + (A-\lambda I)t} \\ &= e^{(\lambda I)t} e^{(A-\lambda I)t} \\ &= e^{\lambda t} e^{(A-\lambda I)t} \\ &= e^{\lambda t} \sum_{j=0}^{k-1} (A - \lambda I)^j (t^j / j!) \end{aligned}$$

$\Rightarrow$ only k terms of exponential series required

Example

Ex.: $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$: $T = -2,\ D = 1$

$$\begin{aligned} p(\lambda) &= \lambda^2 - T\lambda + D \\ &= \lambda^2 + 2\lambda + 1 \\ &= (\lambda + 1)^2 \end{aligned}$$

$\Rightarrow$ only one eigenvalue $\lambda = -1$

$$\begin{aligned} A + I &= \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \\ (A + I)^2 &= \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (k = 2) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad e^{(A+I)t} &= I + (A + I)t \\ &= \begin{bmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad e^{At} &= e^{-t} e^{(A+I)t} \\ &= e^{-t} \begin{bmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{bmatrix} \end{aligned}$$

Generalized Eigenvectors

If A has repeated eigenvalues, n linearly independent eigenvectors may not exist $\Rightarrow$ need generalized eigenvectors.

Generalized Eigenvectors: Definition

Def.: Let λ be eigenvalue of A.

(a) The algebraic multiplicity, m, of λ is the multiplicity of λ as root of the characteristic polynomial (CN Sec. 9.5).

(b) The geometric multiplicity, m_g, of λ is $\dim \operatorname{null}(A - \lambda I)$.

Need: m linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$ associated with λ.

- If $m_g = m \Rightarrow m$ linearly independent eigenvector solutions.
- What if $m_g < m$?

Thm.: If λ is an eigenvalue with algebraic multiplicity m, then there is an integer k, $0 < k \le m$, such that

$$\begin{aligned} \dim \operatorname{null}((A - \lambda I)^k) &= m \\ \dim \operatorname{null}((A - \lambda I)^{k-1}) &< m \end{aligned}$$

Def.: Any nonzero vector $\mathbf{v}$ in $\operatorname{null}((A - \lambda I)^k)$ is a generalized eigenvector for λ.

Solution associated with $\mathbf{v}$*:*

$$(A - \lambda I)^k \mathbf{v} = \mathbf{0} \quad \Rightarrow$$

$$e^{At}\mathbf{v} = e^{\lambda t} \sum_{j=0}^{k-1} (t^j/j!)(A - \lambda I)^j \mathbf{v}$$

Generalized Eigenvectors and Associated Solutions

Thm.: Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be a basis of null$(A - \lambda I)^k$. Then the

$$\mathbf{x}_i(t) = e^{\lambda t} \sum_{j=0}^{k-1} (t^j/j!)(A - \lambda I)^j \mathbf{v}_i,$$

$1 \le i \le m$, are m linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$.

Example: 2d Systems

2d Systems: (Sec. 9.2)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \left\{ \begin{array}{rcl} T & = & a+d \\ D & = & ad-bc \end{array} \right\}$$

Assume $T^2 - 4D = 0 \Rightarrow$

$p(\lambda) = (\lambda - \lambda_1)^2, \quad \lambda_1 = T/2$

(a) If $A = \lambda_1 I \Rightarrow m_g = 2$

$$\Rightarrow \mathbf{x}(t) = e^{\lambda_1 t}\mathbf{x}(0)$$

(any vector is eigenvector)

(b) If $A \neq \lambda_1 I \Rightarrow m_g = 1$:

- Compute eigenvector $\mathbf{v}$
- Pick vector $\mathbf{w}$ that is *not* a multiple of $\mathbf{v}$

  $$\Rightarrow (A - \lambda_1 I)\mathbf{w} = a\mathbf{v}$$

  for some $a \neq 0$ (any $\mathbf{w} \in \mathbf{R}^2$ is generalized eigenvector)

- $\Rightarrow$ F.S.S.:

$$\begin{array}{rcl} \mathbf{x}_1(t) & = & e^{\lambda_1 t}\mathbf{v} \\ \mathbf{x}_2(t) & = & e^{\lambda_1 t}(\mathbf{w} + a\mathbf{v}t) \end{array}$$

Example 1

Ex.: $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$: $T = -2$, $D = 1$

$\Rightarrow T^2 - 4D = 0 \Rightarrow$ eigenvalue $\lambda = -1$

$$A + I = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

$\Rightarrow$ eigenvector $\mathbf{v} = [-2, 1]^T$

Choose $\mathbf{w} = [1, 0]^T$ (simple form) $\Rightarrow$

$$(A+I)\mathbf{w} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\mathbf{v}$$

$\Rightarrow$ F.S.S.:

$$\begin{aligned} \mathbf{x}_1(t) &= e^{-t}\mathbf{v} = e^{-t}[-2, 1]^T \\ \mathbf{x}_2(t) &= e^{-t}(\mathbf{w} - \mathbf{v}t) \\ &= e^{-t}([1, 0]^T - t[-2, 1]^T) \\ &= e^{-t}[1 + 2t, -t]^T \end{aligned}$$

Other Method: Compute (c.f. p.4)

$$e^{At} = e^{-t}(I + (A+I)t) = e^{-t} \begin{bmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{bmatrix}$$

Columns of e^{At} are also F.S.S.

Example 2

Ex.: $A = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & -3 & -3 \end{bmatrix}$

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} -1-\lambda & 2 & 1 \\ 0 & -1-\lambda & 0 \\ -1 & -3 & -3-\lambda \end{vmatrix} \\ &= (-1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ -1 & -3-\lambda \end{vmatrix} \\ &= (-1-\lambda)[(1+\lambda)(3+\lambda)+1] \\ &= -(\lambda+1)(\lambda^2+4\lambda+4) \\ &= -(\lambda+1)(\lambda+2)^2 \end{aligned}$$

$\Rightarrow$ eigenvalues $\lambda_1 = -1,\ m_1 = 1$
$\lambda_2 = -2,\ m_2 = 2$

Compute $A - \lambda_1 I = A + I$:

$$A+I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Set $x_3 = -2$
$\Rightarrow$ eigenvector $\mathbf{v}_1 = [1, 1, -2]^T$

Since $m_1 = 1$
$\Rightarrow$ one (eigenvector) solution:

$$\mathbf{x}_1(t) = e^{-t}[1, 1, -2]^T$$

Example 2 (cont.)

$m_2 = 2 \to$ check $A - \lambda_2 I$, $(A - \lambda_2 I)^2$:

$$A+2I = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A+2I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A + 2I \to$ eigenvector $\mathbf{v}_2 = [1, 0, -1]^T$
$\Rightarrow$ eigenvector solution:

$$\mathbf{x}_2(t) = e^{-2t}[1, 0, -1]^T$$

For $\mathbf{x}_3(t)$ use generalized eigenvector $\mathbf{v}_3$ that is *not* an eigenvector.

Basis of null$((A+2I)^2)$: $\begin{cases} \mathbf{u}_1 = [1, 0, 0]^T \\ \mathbf{u}_2 = [0, 0, 1]^T \end{cases}$

Note: $\mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_2 = (A + 2I)\mathbf{u}_1 = (A + 2I)\mathbf{u}_2$

$\mathbf{v}_3$ can be any vector $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ s.t. $(A+2I)(c_1\mathbf{u}_1+c_2\mathbf{u}_2) = (c_1+c_2)\mathbf{v}_2 \neq \mathbf{0}$

Choose $\mathbf{v}_3 = \mathbf{u}_2 = [0, 0, 1]^T$ (text: $\mathbf{u}_1$)

$$\begin{aligned} \Rightarrow \quad \mathbf{x}_3(t) &= e^{-2t}(I\mathbf{v}_3 + t(A + 2I)\mathbf{v}_3) \\ &= e^{-2t}(\mathbf{v}_3 + t\mathbf{v}_2) \\ &= e^{-2t}[t, 0, 1 - t]^T \end{aligned}$$

7

Example 3

Ex.: $A = \begin{bmatrix} 6 & 6 & -3 & 2 \\ -4 & -4 & 2 & 0 \\ 8 & 7 & -4 & 4 \\ 1 & 0 & -1 & -2 \end{bmatrix}$

Matlab $\to p(\lambda) = ((\lambda+1)^2+1)^2$

$\Rightarrow$ single complex pair of eigenvalues

$\lambda_1 = -1+i,\ \lambda_2 = \overline{\lambda_1} \quad (m=2).$

1. Check $B \equiv A - \lambda_1 I = A - (-1+i)I$:

$$B = \begin{bmatrix} 7-i & 6 & -3 & 2 \\ -4 & -3-i & 2 & 0 \\ 8 & 7 & -3-i & 4 \\ 1 & 0 & -1 & -1-i \end{bmatrix}$$

Matlab $\to$ basis for null(B):

$$\mathbf{v}_1 = [2, 0, 4, -1+i]^T$$

$\Rightarrow$ Complex eigenvector solution:

$$\mathbf{z}_1(t) = e^{(-1+i)t}[2, 0, 4, -1+i]^T$$

2. Check $B^2 = (A - \lambda_1 I)^2$:

$$B^2 = \begin{bmatrix} 2-14i & 3-12i & -2+6i & -4i \\ 8i & -2+6i & -4i & 0 \\ 8-16i & 6-14i & -6+6i & -8i \\ -2-2i & -1 & 1+2i & -2+2i \end{bmatrix}$$

Matlab $\to$ basis for null(B^2):

$$\begin{aligned} \mathbf{u}_1 &= [2, 0, 4, -1+i]^T = \mathbf{v}_1 \\ \mathbf{u}_2 &= [-3-i, 4, 0, -2+2i]^T \end{aligned}$$

$\Rightarrow \mathbf{u}_2$ is generalized eigenvector that is not an eigenvector.

Pick $\mathbf{v}_3 = \mathbf{u}_2 = [-3-i, 4, 0, -2+2i]^T$

Need: $B\mathbf{v}_3 = [-2, 0, -4, 1-i]^T = -\mathbf{v}_2$

Complex solution associated with $\mathbf{v}_3$:

$$\begin{aligned} \mathbf{z}_2(t) &= e^{(-1+i)t}(I\mathbf{v}_3 + tB\mathbf{v}_3) \\ &= e^{(-1+i)t}(\mathbf{v}_3 - t\mathbf{v}_2) \\ &= e^{(-1+i)t}[-3-i-2t, 4, -4t, \\ &\qquad -2+2i+(1-i)t]^T \end{aligned}$$

Example 3 (cont.)

3. Take real and imaginary parts of $\mathbf{z}_1(t)$ and $\mathbf{z}_2(t)$ to obtain F.S.S:

$\mathbf{x}_1(t) = \operatorname{Re}\mathbf{z}_1(t) = e^{-t}[2\cos t, 0, 4\cos t, -\cos t - \sin t]^T$

$\mathbf{x}_2(t) = \operatorname{Im}\mathbf{z}_1(t) = e^{-t}[2\sin t, 0, 4\sin t, \cos t - \sin t]^T$

$\mathbf{x}_3(t) = \operatorname{Re}\mathbf{z}_2(t) = e^{-t}[\sin t - (3+2t)\cos t, 4\cos t, -4t\cos t, (t-2)(\cos t + \sin t)]^T$

$\mathbf{x}_4(t) = \operatorname{Im}\mathbf{z}_2(t) = e^{-t}[-\cos t - (3+2t)\sin t, -4\sin t, -4t\sin t, (t-2)(\sin t - \cos t)]^T$

Example 4

Ex.: $A = \begin{bmatrix} 7 & 5 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 12 & 10 & -5 & 4 \\ -4 & -4 & 2 & -1 \end{bmatrix}$

Matlab $\to p(\lambda) = (\lambda+1)(\lambda-1)^3$

$\Rightarrow$ eigenvalues $\lambda_1 = -1,\ m_1 = 1$
$\lambda_2 = 1,\ m_2 = 3$

Find eigenvector for λ_1:

$$A + I = \begin{bmatrix} 8 & 5 & -3 & 2 \\ 0 & 2 & 0 & 0 \\ 12 & 10 & -4 & 4 \\ -4 & -4 & 2 & 0 \end{bmatrix}$$

Matlab $\to$ eigenvector (basis vector for null$(A+I)$): $\mathbf{v}_1 = [1, 0, 2, -1]^T$

Associated eigenvector solution:

$$\mathbf{x}_1(t) = e^{-t}[1, 0, 2, -1]^T$$

For $\lambda_2 = 1 \to$ check powers of $A - I$:

$$B \equiv A - I = \begin{bmatrix} 6 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 12 & 10 & -6 & 4 \\ -4 & -4 & 2 & -2 \end{bmatrix}$$

Matlab $\to$ basis of null(B):

$$\begin{aligned} \mathbf{v}_2 &= [1, 0, 2, 0]^T \\ \mathbf{v}_3 &= [1, -2, 0, 2]^T \end{aligned}$$

Associated eigenvector solutions:

$$\begin{aligned} \mathbf{x}_2(t) &= e^{t}[1, 0, 2, 0]^T \\ \mathbf{x}_3(t) &= e^{t}[1, -2, 0, 2]^T \end{aligned}$$

Example 2 (cont.)

To find 4th solution check B^2:

$$B^2 = \begin{bmatrix} -8 & -8 & 4 & -4 \\ 0 & 0 & 0 & 0 \\ -16 & -16 & 8 & -8 \\ 8 & 8 & -4 & 4 \end{bmatrix}$$

$\Rightarrow$ $RREF(B^2)$ has only one nonzero row $[1, 1, -1/2, 1/2]$.

Construct basis of null(B^2) by setting
$x_2, x_3 = 0, x_4 = 2 \rightarrow \mathbf{u}_1 = [-1, 0, 0, 2]^T$
$x_2, x_4 = 0, x_3 = 2 \rightarrow \mathbf{u}_2 = [1, 0, 2, 0]^T$
$x_3, x_4 = 0, x_2 = 1 \rightarrow \mathbf{u}_3 = [1, -1, 0, 0]^T$
Check which are *not* eigenvectors:

$$B\mathbf{u}_1 = -2\mathbf{v}_2, \; B\mathbf{u}_2 = \mathbf{0}, \; B\mathbf{u}_3 = \mathbf{v}_2$$

$\Rightarrow$ Can choose $\mathbf{v}_4 = \mathbf{u}_1$ (simple).

Associated solution:

$$\begin{aligned} \mathbf{x}_4(t) &= e^t(I\mathbf{v}_4 + tB\mathbf{v}_4) = e^t(\mathbf{u}_1 - 2t\mathbf{v}_2) \\ &= e^t[-1 - 2t, 0, -4t, 2]^T \end{aligned}$$