Probability

First Exam **MATH 3338-10853 (Fall 2006)** **September 13, 2006**

This exam has 2 questions, for a total of 0 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name and UH-ID: ______________________________

1. (a) Let $x_1, \ldots, x_n$ be a sample and c be a constant. Prove that

   1. if $y_1 = x_1 + c, \ldots, y_n = x_n + c$, then $s_y^2 = s_x^2$, and
   2. if $y_1 = c\,x_1, \ldots, y_n = c\,x_n$, then $s_y^2 = c^2 s_x^2$,

   where s_x^2 is the sample variance of the x's and s_y^2 is the sample variance of the y's.

   (b) The article "A Thin-Film Oxygen Uptake Test for the Evaluation of Automotive Crankcase Lubricants" reported the following data on oxidation-induction time (min) for various commercial oils:

   87 103 130 160 180 195 132 145 211 105 145 153 152 138 87 99 93 119 129

   1. Calculate the sample variance and standard deviation.
   2. If the observations were re-expressed in hours, what would be the resulting values of the sample variance and sample standard deviation? Answer without actually performing the re-expression.

Solution:

(a) 1. First,

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(x_i + c) = \frac{1}{n}\sum_{i=1}^{n} x_i \; + \; c = \bar{x} + c.$$

Then

$$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i + c - (\bar{x} + c)\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = s_x^2.$$

2. First,

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(c\,x_i) = c\,\frac{1}{n}\sum_{i=1}^{n} x_i = c\,\bar{x}.$$

Then

$$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(c\,x_i - c\,\bar{x}\right)^2 = c^2\,\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = c^2\,s_x^2.$$

(b) 1. First compute

$$\begin{aligned}\sum x_i &= 87 + 103 + 130 + 160 + 180 + 195 + 132 + 145 + 211 + 105 \\ &\quad + 145 + 153 + 152 + 138 + 87 + 99 + 93 + 119 + 129 \\ &= 2563\end{aligned}$$

$$\begin{aligned}\sum x_i^2 &= 87^2 + 103^2 + 130^2 + 160^2 + 180^2 + 195^2 + 132^2 + 145^2 + 211^2 + 105^2 \\ &\quad + 145^2 + 153^2 + 152^2 + 138^2 + 87^2 + 99^2 + 93^2 + 119^2 + 129^2 \\ &= 368501\end{aligned}$$

Then

$$s_x^2 = \frac{1}{n-1}\left(\sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2\right) = (368501 - 2563^2/19)/18 \approx 1264.8$$

and

$$s_x \approx 35.564$$

2. If y_i are the observations reexpressed in hours, then

$$y_i = cx_i \quad \text{with } c = 1/60.$$

So

$$s_y^2 = c^2 s_x^2 \approx 1264.8/60^2 \approx 0.35133$$

and

$$s_y = cs_x \approx 35.564/60 \approx 0.59273$$

2. (a) Prove that for any events A and B, we have

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

(these are called De Morgan's laws)

(b) Use Venn diagrams to verify De Morgan's laws.

Solution:

(a) 1. First, show that $(A \cup B)' \subset A' \cap B'$.

$$\forall x \in (A \cup B)' \Rightarrow x \notin A \cup B \Rightarrow x \notin A \text{ and } x \notin B \Rightarrow x \in A' \cap B'$$

Then, show that $(A \cup B)' \supset A' \cap B'$.

$$\forall x \in A' \cap B' \Rightarrow x \notin A \text{ and } x \notin B \Rightarrow x \notin A \cup B \Rightarrow x \in (A \cup B)'$$

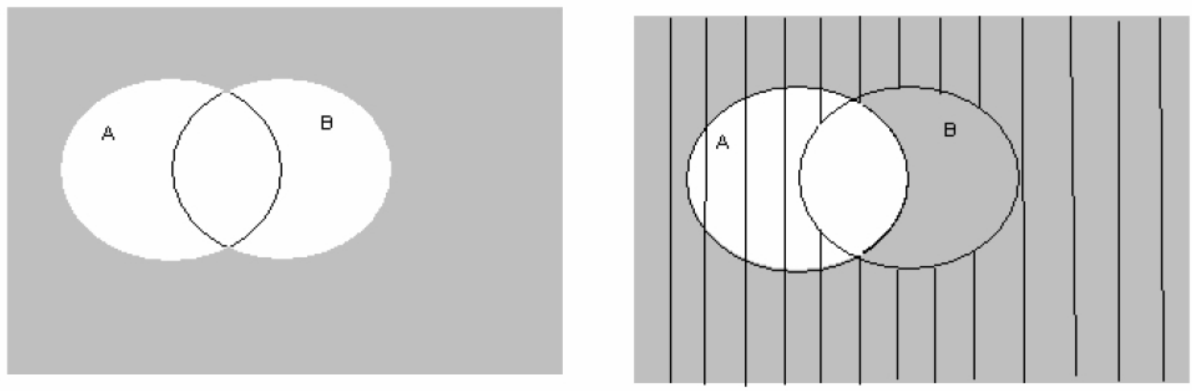
2. First, show that $(A \cap B)' \subset A' \cup B'$.

$$\forall x \in (A \cap B)' \Rightarrow x \notin A \cap B \Rightarrow x \notin A \text{ or } x \notin B \Rightarrow x \in A' \cup B'$$

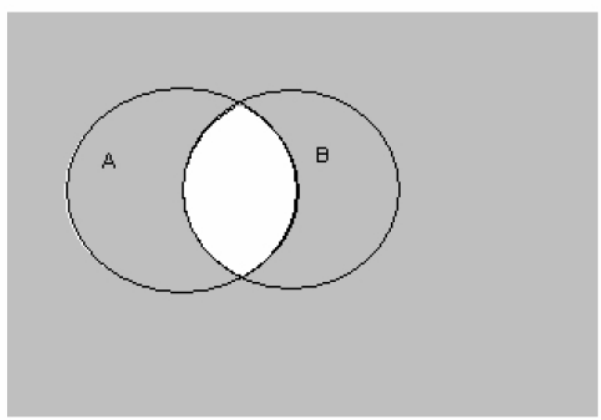
Then, show that $(A \cap B)' \supset A' \cup B'$.

$$\forall x \in A' \cup B' \Rightarrow x \notin A \text{ or } x \notin B \Rightarrow x \notin A \cap B \Rightarrow x \in (A \cap B)'$$

a. In the diagram on the left, the shaded area is $(A \cup B)'$. On the right, the shaded area is A', the striped area is B', and the intersection $A' \cap B'$ occurs where there is BOTH shading and stripes. These two diagrams display the same area.



b. In the diagram below, the shaded area represents $(A \cap B)'$. Using the diagram on the right above, the union of A' and B' is represented by the areas that have either shading or stripes or both. Both of the diagrams display the same area.



When you finish this exam, you should go back and reexamine your work for any errors that you may have made.