This exam has 3 questions, for a total of 100 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name and UH-ID: ______________________

1. In flipping a fair coin let X be the number of tosses to get the first head. Then $p(x) = .5^x$ for $x = 1, 2, 3, \ldots$. Find the moment generating function $M_X(t)$ and use it to get the mean $E(X)$ and the variance $V(X)$.

Solution: Note that X is a geometric rv with $P(\text{head}) = 0.5$, thus the pmf of X is

$$p(k) = p(1-p)^{k-1} = 0.5^k, \quad \text{for } k = 1, 2, 3, \ldots.$$

Then, the mgf of X is

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \sum_x e^{xt} p(x) = \sum_{k=1}^{\infty} e^{kt} p(1-p)^{k-1} = pe^t \sum_{k=1}^{\infty} [(1-p)e^t]^{k-1} \\ &= \frac{pe^t}{1-(1-p)e^t} \left(= \frac{0.5e^t}{1-0.5e^t} = \frac{e^t}{2-e^t} \right). \end{aligned}$$

Using the rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, we differentiate $M_X(t)$

$$\begin{aligned} M_X^{(1)}(t) &= \frac{pe^t[1-(1-p)e^t] - pe^t[-(1-p)e^t]}{[1-(1-p)e^t]^2} = \frac{pe^t}{[1-(1-p)e^t]^2} \\ M_X^{(2)}(t) &= \frac{pe^t[1-(1-p)e^t]^2 - pe^t[1-(1-p)e^t][-2(1-p)e^t]}{[1-(1-p)e^t]^4} = \frac{pe^t[1+(1-p)e^t]}{[1-(1-p)e^t]^3} \end{aligned}$$

Setting $t = 0$ gives

$$\begin{aligned} E(X) &= M_X^{(1)}(t=0) = \left.\frac{pe^t}{[1-(1-p)e^t]^2}\right|_{t=0} = \frac{p}{[1-(1-p)]^2} = \frac{1}{p}, \\ E(X^2) &= M_X^{(2)}(t=0) = \left.\frac{pe^t[1+(1-p)e^t]}{[1-(1-p)e^t]^3}\right|_{t=0} = \frac{p[1+(1-p)]}{[1-(1-p)]^3} = \frac{2-p}{p^2}, \\ V(X) &= E(X^2) - [E(X)]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}. \end{aligned}$$

Thus, with $p = 0.5$, we have the mean and variance of X

$$E(X) = \frac{1}{0.5} = 2, \quad V(X) = \frac{1-0.5}{0.5^2} = 2.$$

2. The College Board reports that 2% of the 2 million high school students who take the SAT each year receive special accommodations because of documented disabilities. Consider a random sample of 25 students who have recently taken the test

   a. What is the probability that exactly 1 received a special accommodation?

   b. What is the probability that at least 1 received a special accommodation?

   c. What is the probability that at least 2 received a special accommodation?

   d. What is the probability that the number among the 25 who received a special accommodation is within 2 standard deviations of the number you would expect to be accommodated?

   Solution: We have $p = P(\text{a student received a special accommodation}) = 0.02$, so with X = the number among the 25 who received a special accommodation, $X \sim \text{Bin}(25, 0.02)$.

   a. The probability that exactly 1 received a special accommodation is

   $$P(X=1) = \binom{25}{1}(0.02)^1(1-0.02)^{25-1} = 25(0.02)(0.98)^{24} \approx 0.3079$$

   b. The probability that at least 1 received a special accommodation is

   $$\begin{aligned} P(X \geq 1) &= 1 - P(X=0) = 1 - \binom{25}{0}(0.02)^0(1-0.02)^{25} \\ &= 1 - (0.98)^{25} \approx 1 - 0.6035 = 0.3965 \end{aligned}$$

   c. The probability that at least 2 received a special accommodation is

   $$P(X \geq 2) = 1 - P(X=0) - P(X=1) \approx 1 - 0.6035 - 0.3079 = 0.0886$$

   d. The mean and the standard deviation of X are

   $$\begin{aligned} \mu &= E(X) = np = 25(0.02) = 0.5, \\ \sigma &= \sqrt{V(X)} = \sqrt{np(1-p)} = \sqrt{25(0.02)(0.98)} = \sqrt{0.49} = 0.7. \end{aligned}$$

   Then, the probability that the number among the 25 who received a special accommodation is within 2 standard deviations of the expected number is

   $$\begin{aligned} P(|X-\mu| \leq 2\sigma) &= P(X \leq \mu + 2\sigma) = P(X \leq 0.5 + (2)(0.7)) = P(X \leq 1.9) \\ &= P(X=0) + P(X=1) \approx 0.6035 + 0.3079 = 0.9114. \end{aligned}$$

3. Show that the mean of a binomial rv X having pmf $b(x; n, p)$ is $E(X) = np$.

Solution:

$$\begin{aligned}
E(X) &= \sum_x xp(x) = \sum_{k=0}^{n} b(k; n, p) = \sum_{k=0}^{n} k \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-k)!} p^l (1-p)^{(n-1)-l} = np[p + (1-p)]^{n-1} = np.
\end{aligned}$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.