This exam has 3 questions, for a total of 100 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Name and UH-ID: ______________________________

1. Commuting to work requiring getting on a bus near home and then transferring to a second bus. If the waiting time (in minutes) at each stop has a uniform distribution with $A = 0$ and $B = 5$, then it can be shown that the total waiting time Y has the pdf

$$f(y) = \begin{cases} \frac{1}{25}y & 0 \le y < 5 \\ \frac{2}{5} - \frac{1}{25}y & 5 \le y \le 10 \\ 0 & y < 0 \text{ or } y > 10 \end{cases}$$

(a) Sketch a graph of the pdf of Y.

(b) Verify that $\int_{-\infty}^{\infty} f(y)dy = 1$.

(c) Compute and sketch the cdf of Y.

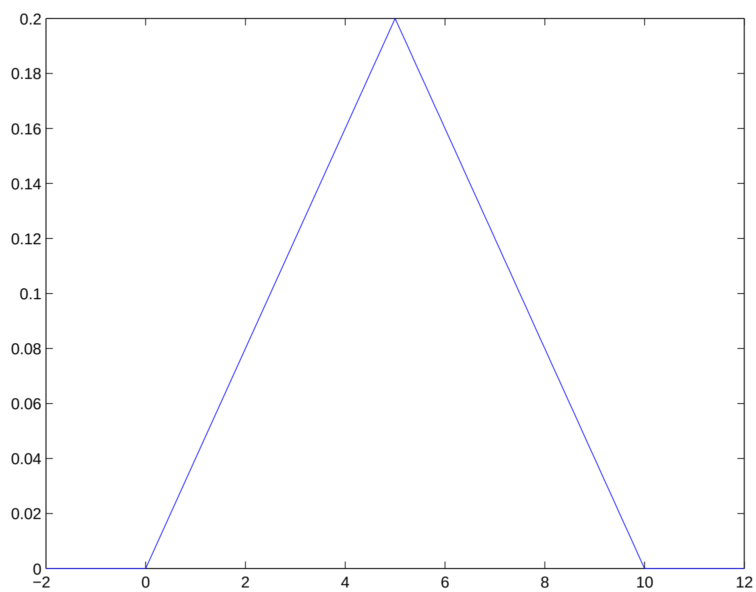
(d) What is the probability that total waiting time is between 3 and 8 minutes?

(e) Compute $E(Y)$ and $V(Y)$. How do these compare with the expected waiting time and variance for a single bus when the time is uniformly distributed on $[0, 5]$?

(f) Explain how symmetry can be used to obtain $E(Y)$.

Solution:

(a) The graph of the pdf of Y is



(b) We have $f(y) \leq 0$ for any y, and

$$\int_{-\infty}^{\infty} f(y)dy = \int_0^5 \tfrac{1}{25}ydy + \int_5^{10} \left(\tfrac{2}{5} - \tfrac{1}{25}y\right) dy = \left[\tfrac{1}{50}y^2\right]_0^5 + \left[\tfrac{2}{5}y - \tfrac{1}{50}y^2\right]_5^{10}$$
$$= \tfrac{1}{2} + \left((4-2) - (2 - \tfrac{1}{2})\right) = \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

(c) For $0 \leq y < 5$,

$$F(y) = \int_{-\infty}^{y} f(u)du = \int_0^y \frac{1}{25}udu = \left[\frac{1}{50}u^2\right]_0^y = \frac{1}{50}y^2;$$

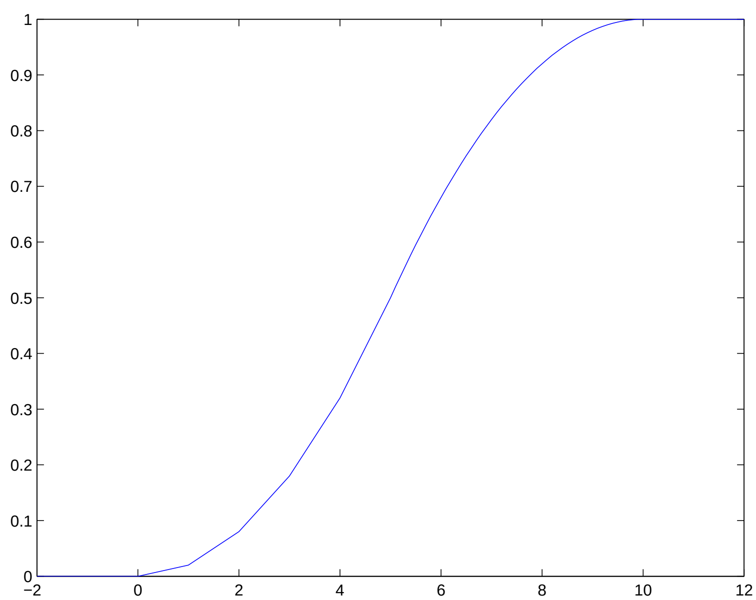
for $5 \leq y < 10$,

$$F(y) = \int_{-\infty}^{y} f(u)du = \int_0^5 \tfrac{1}{25}udu + \int_5^y \left(\tfrac{2}{5} - \tfrac{1}{25}u\right) du = \left[\tfrac{1}{50}u^2\right]_0^5 + \left[\tfrac{2}{5}u - \tfrac{1}{50}u^2\right]_5^y$$
$$= \tfrac{1}{2} + \left((\tfrac{2}{5}y - \tfrac{1}{50}y^2) - (2 - \tfrac{1}{2})\right) = \tfrac{2}{5}y - \tfrac{1}{50}y^2 - 1.$$

Then the cdf of Y is

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{50}y^2 & 0 \leq y < 5 \\ \frac{2}{5}y - \frac{1}{50}y^2 - 1 & 5 \leq y \leq 10 \\ 1 & y > 10 \end{cases}$$

And the graph of the cdf of Y is



(d) The probability that total waiting time is between 3 and 8 minutes is

$$P(3 \leq Y \leq 8) = F(8) - F(3) = \left(\frac{2}{5}8 - \frac{1}{50}8^2 - 1\right) - \left(\frac{1}{50}3^2\right) = \frac{46}{50} - \frac{9}{50} = \frac{37}{50} = 0.74$$

(e) The expected waiting time and variance of Y are

$E(Y) = \int_{-\infty}^{\infty} y f(y) dy = \int_0^5 \frac{1}{25} y^2 dy + \int_5^{10} \left(\frac{2}{5}y - \frac{1}{25}y^2\right) dy = \left[\frac{1}{75}y^3\right]_0^5 + \left[\frac{2}{10}y^2 - \frac{1}{75}y^3\right]_5^{10}$
$= \frac{5}{3} + \left((20 - \frac{40}{3}) - (5 - \frac{5}{3})\right) = 5.$

$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y) dy = \int_0^5 \frac{1}{25} y^3 dy + \int_5^{10} \left(\frac{2}{5}y^2 - \frac{1}{25}y^3\right) dy = \left[\frac{1}{100}y^4\right]_0^5 + \left[\frac{2}{15}y^3 - \frac{1}{100}y^4\right]_5^{10}$
$= \frac{25}{4} + \left((\frac{400}{3} - 100) - (\frac{50}{3} - \frac{25}{4})\right) = \frac{350}{12}$

$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{350}{12} - 5^2 = \frac{50}{12} \approx 4.1667$

Let X be the waiting time and variance for a single bus. Then the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{5} & 0 \le x \le 5 \\ 0 & x < 0 \text{ or } x > 5 \end{cases}$$

Then the expected waiting time and variance of X are

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^5 \frac{1}{5} x dx = \left[\frac{1}{10}y^2\right]_0^5 = \frac{5}{2} = 2.5$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^5 \frac{1}{5} x^2 dx = \left[\frac{1}{15}y^3\right]_0^5 = \frac{25}{3}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{25}{3} - \left(\frac{5}{2}\right)^2 = \frac{25}{12} \approx 2.0833.$$

Thus $E(Y) = 2E(X)$ and $V(Y) = V(X)$.

(f) Since the pdf of Y is symmetric about 5, obviously $E(Y) = 5$.

2. A result called **Chebyshev's inequality** states that for any probability distribution of an rv X and any number k that is at least 1, $P(|X-\mu| \geq k\sigma) \leq 1/k^2$. In words, the probability that the value of X lies at least k standard deviations from its mean is at most $1/k^2$.

   (a) What is the value of the upper bound of Chebyshev's inequality for $k = 1$, 2, and 3?

   (b) Obtain this probability in the case of X having a normal distribution for $k = 1$, 2, and 3, and compare to the upper bound of Chebyshev's inequality. Appendix Table A.3 of Textbook contains the values $\Phi(-3) = 0.0013$, $\Phi(-2) = 0.0228$, $\Phi(-1) = 0.1587$, $\Phi(1) = 0.8413$, $\Phi(2) = 0.9772$, $\Phi(3) = 0.9987$.

Solution:

(a) The value of the upper bound of Chebyshev's inequality $1/k^2$ for $k = 1$, 2, and 3 are

$$1/1^2 = 1, \quad 1/2^2 = 1/4 = 0.25, \quad 1/3^2 = 1/9 \approx 0.11$$

(b) In the case of X having a normal distribution with mean μ and standard deviation σ, we define $Z = \frac{X-\mu}{\sigma}$. Then, Z has the standard normal distribution, and we obtain the probability

$$P(|X-\mu| \geq k\sigma) = 1 - P(|X-\mu| \leq k\sigma) = 1 - P(-k\sigma \leq X-\mu \leq k\sigma)$$

$$= 1 - P(-k \leq \tfrac{X-\mu}{\sigma} \leq k) = 1 - P(-k \leq Z \leq k) = 1 - (P(Z \leq k) - P(Z \leq -k))$$

$$= 1 - (\Phi(k) - \Phi(-k)) = \begin{cases} 1 - (\Phi(1) - \Phi(-1)) = 0.3174 & \text{for } k = 1 \\ 1 - (\Phi(2) - \Phi(-2)) = 0.0456 & \text{for } k = 2 \\ 1 - (\Phi(3) - \Phi(-3)) = 0.0026 & \text{for } k = 3 \end{cases}$$

These values are considerably less than the bounds 1, 0.25, and 0.11 given by Chebyshev's inequality.

3. Let X have a normal distribution with mean μ and standard deviation σ.

   (a) Find the pdf of $Y = e^X$. The distribution of Y is lognormal.

   (b) Find the pdf of $Z = X^2$. The distribution of Z is chi-squared with 1 degree of freedom for $\mu = 0$ and $\sigma = 1$.

Solution:

(a) The rv X has a normal distribution with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

From the relation $Y = e^X$, we have $y > 0$ and $y = e^x$, which is increasing and has the inverse $x = \ln y$. Then the pdf of Y is

$$f_Y(y) = f_X(\ln y)\frac{d}{dy}(\ln y) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(\ln y-\mu)^2\right)\left(\frac{1}{y}\right)$$
$$= \frac{1}{\sqrt{2\pi}\sigma y}\exp\left(-\frac{1}{2\sigma^2}(\ln y-\mu)^2\right)$$

Thus the distribution of Y is lognormal.

(b) Note that the function $z = x^2$ is not one-to-one. The cdf of Z is given by

$$F_Z(z) = P(Z \le z) = P(X^2 \le z) = P(-\sqrt{z} \le X \le \sqrt{z}) = F_X(\sqrt{z}) - F_X(-\sqrt{z}).$$

Thus Z has the pdf:

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{d}{dz}\Big(F_X(\sqrt{z}) - F_X(-\sqrt{z})\Big) = \frac{1}{2\sqrt{z}}\Big(f_X(\sqrt{z}) + f_X(-\sqrt{z})\Big).$$

The pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Then we have

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{2\sqrt{z}}\left(e^{-\frac{(\sqrt{z}-\mu)^2}{2\sigma^2}} + e^{-\frac{(-\sqrt{z}-\mu)^2}{2\sigma^2}}\right)$$

For $\mu = 0$ and $\sigma = 1$, Z has the pdf

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{z}}e^{-\frac{z}{2}}$$

thus has the chi-squared distribution with 1 degree of freedom

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.