Review

2.4 Conditional Probability.

* Tree Diagrams: to illustrate what is going on when the experiment consists of several stages in succession, and to keep track of all the possibilities.
  - Any edge (branch) corresponds to an event, which is indicated at the appropriate node or vertex. The relevant probability is written adjacent to the edge. The final node is referred to as a leaf. The probability of the event at any node, or leaf, is obtained by multiplying the probabilities labelling the branches leading to it. (multiplication rule)

    For example:

    - $$\Omega \xrightarrow{P(A)} A \xrightarrow{P(B|A)} A \cap B$$

multiplication rule

$$P(A \cap B) = P(B|A)\, P(A)$$

- $$\Omega \xrightarrow{P(A_1)} A_1 \xrightarrow{P(A_2|A_1)} A_1 \cap \cancel{A_2} \xrightarrow{P(A_3|A_1 \cap A_2)} A_1 \cap A_2 \cap A_3$$

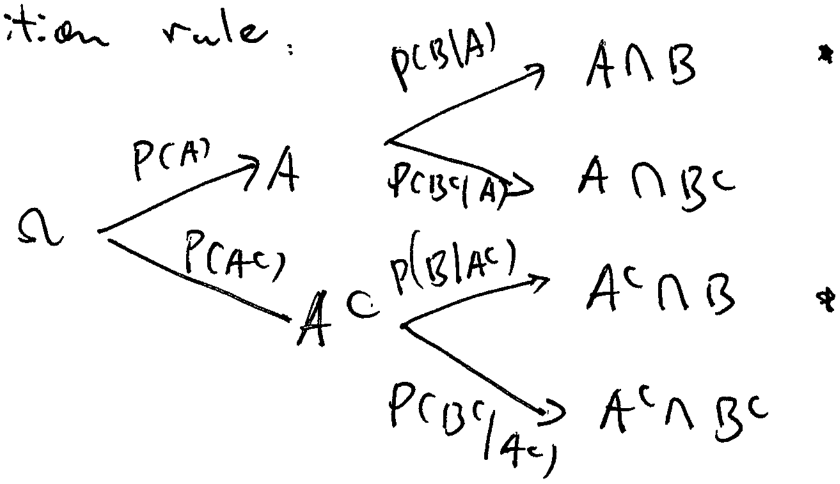
$$P(A_1 \cap A_2 \cap A_3) = P(A_3|A_1 \cap A_2)\, P(A_2|A_1)\, P(A_1)$$

- $$\Omega \xrightarrow{P(A_1)} A_1 \xrightarrow{P(A_2|A_1)} A_1 \cap A_2 \rightarrow \cdots \xrightarrow{P(A_n|A_1 \cap \cdots \cap A_{n-1})} A_1 \cap A_2 \cap \cdots \cap A_n$$

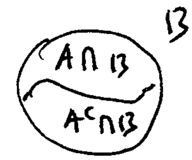
$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_n | A_1 \cap \cdots \cap A_{n-1}) \cdots$$
$$\cdots P(A_2|A_1)\, P(A_1)$$

— Partition rule:

∘



Since event B occurs at two leaves marked with an asterisk.



Partition: $B = A \cap B + A^c \cap B$

the diagram shows that

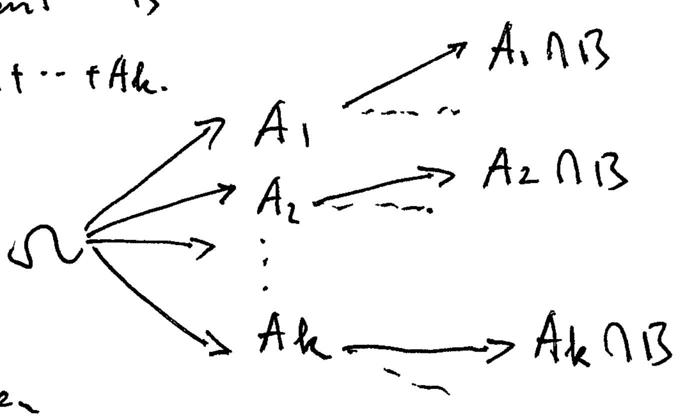
The law of total probability

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

∘ Let $A_1, \dots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B

$\Omega = A_1 + \dots + A_k$.

Partition of B



Then

$$P(B) = P(B|A_1)P(A_1) + \dots + P(B|A_k)P(A_k)$$

* Example 2.29.

$$\begin{tikzcd}[column sep=large]
 & & A_1 \cap B \quad : \quad P(B\cap A_1) = .25 * .5 = .125 \\
 & A_1 \arrow[ur, "P(B|A_1)=.25"] \arrow[r, "P(B^c|A_1)=.75"'] & A_1 \cap B^c \\
 & & A_2 \cap B \quad : \quad P(B\cap A_2) = .2 * .3 = .06 \\
\Omega \arrow[uur, "P(A_1)=0.5"] \arrow[r, "P(A_2)=0.3"] \arrow[ddr, "P(A_3)=0.2"'] & A_2 \arrow[ur, "P(B|A_2)=.2"] \arrow[r, "P(B^c|A_2)=.8"'] & A_2 \cap B^c \\
 & & A_3 \cap B \quad : \quad P(B\cap A_3) = .1 * .2 = .02 \\
 & A_3 \arrow[ur, "P(B|A_3)=.1"] \arrow[r, "P(B^c|A_3)=.9"'] & A_3 \cap B^c
\end{tikzcd}$$

$$P(B) = .125 + .06 + .02 = .205$$

* Reversed Tree Diagrams:

Given a tree diagram

prior information

$$\begin{tikzcd}[column sep=large]
 & & A\cap B \\
 & A \arrow[ur, "P(B|A)"] \arrow[r, "P(B^c|A)"'] & A\cap B^c \\
\Omega \arrow[ur, "P(A)"] \arrow[dr, "P(A^c)"'] & & A^c\cap B \\
 & A^c \arrow[ur, "P(B|A^c)"] \arrow[r, "P(B^c|A^c)"'] & A^c\cap B^c
\end{tikzcd}$$

$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$

$P(B^c) = P(B^c|A)P(A) + P(B^c|A^c)P(A^c)$

We can draw the reversed tree diagram

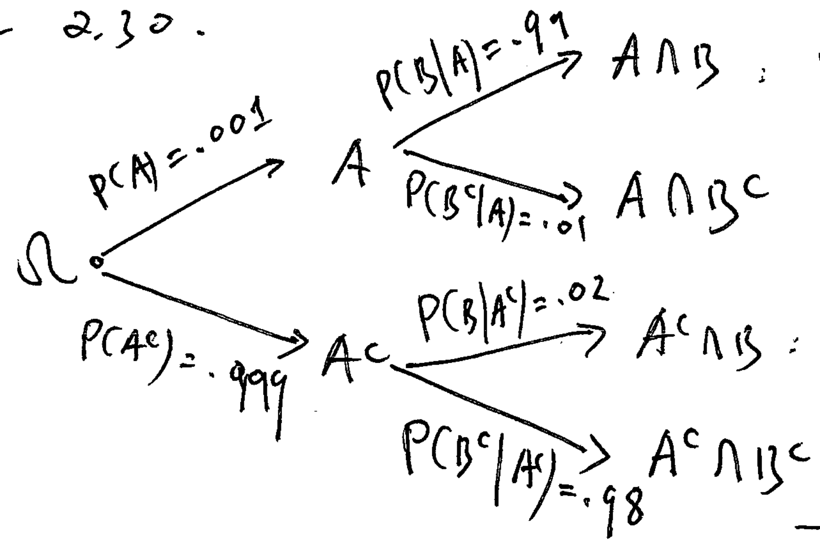
posterior information

$$\begin{tikzcd}[column sep=large]
 & & A\cap B \\
 & B \arrow[ur, "P(A|B)"] \arrow[r, "P(A^c|B)"'] & A^c\cap B \\
\Omega \arrow[ur, "P(B)"] \arrow[dr, "P(B^c)"'] & & A\cap B^c \\
 & B^c \arrow[ur, "P(A|B^c)"] \arrow[r, "P(A^c|B^c)"'] & A^c\cap B^c
\end{tikzcd}$$

If we have the entries on either tree we can find the entries on the other by using Bayes' theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{P(A\cap B)}{P(B)}$$

* Example 2.30.

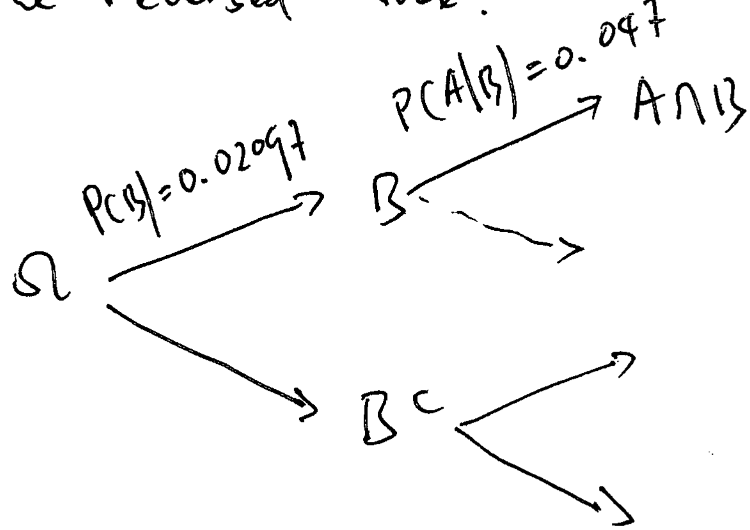
- 

$P(A\cap B) = .001 * .99 = .00099$

$P(A^c\cap B) = .02 * .999 = 0.01998$

$P(B) = 0.00099 + 0.01998 = 0.02097$

- the reversed tree.



$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0.00099}{0.02097} = 0.047$$

2.5 Independence

* Product rule. Events A and B are said to be <u>independent</u> if and only if

$$P(A \cap B) = P(A)\,P(B)$$

* Proposition: Let A and B be independent. Then so are the following pairs of events: (1) A^c and B (2) A and B^c, (3) A^c and B^c.

Proof (1).

$$\begin{aligned} P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \\ &= (1 - P(A))\,P(B) \\ &= P(A^c)\,P(B) \end{aligned}$$

Then A^c and B are independent

(2)

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)\,[1 - P(B)] \\ &= P(A)\,P(B^c) \end{aligned}$$

Then A and B^c are independent.

(3)

$$\begin{aligned} P(A^c \cap B^c) &= P(A^c) - P(A^c \cap B) \\ &= P(A^c) - P(A^c)\,P(B) \\ &= P(A^c)\,[1 - P(B)] \\ &= P(A^c)\,P(B^c) \end{aligned}$$

Then A^c and B^c are independents.

* Proposition. Let A and B be independent. Then

$$(1) \quad P(A) = P(A|B) = P(A|B^c)$$

$$(2) \quad P(B) = P(B|A) = P(B|A^c)$$

Proof. $P(A\cap B) = P(A|B)\,P(B) = P(B|A)\,P(A)$

$$P(A\cap B) = P(A)\cdot P(B).$$

$$\Rightarrow \quad P(A) = P(A|B), \quad P(B) = P(B|A).$$

$$P(A\cap B^c) = P(A|B^c)\,P(B^c)$$

$$P(A\cap B^c) = P(A)\,P(B^c)$$

$$\Rightarrow \quad P(A) = P(A|B^c)$$

$$P(A^c\cap B) = P(B|A^c)\,P(A^c)$$

$$P(A^c\cap B) = P(A^c)\,P(B)$$

$$\Rightarrow \quad P(B) = P(B|A^c).$$

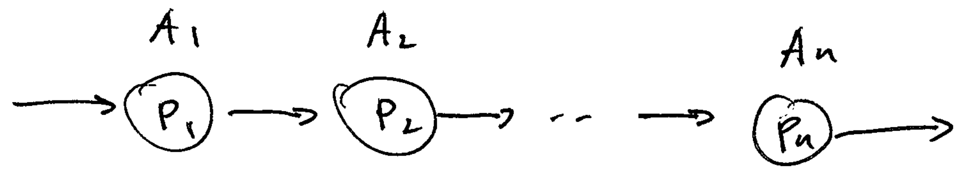
* Independent events: The events $A_1, \dots, A_n$ are independent if and only if

$$P(A_{i_1}\cap A_{i_2}\cap \dots \cap A_{i_k}) = P(A_{i_1})\cdots P(A_{i_k})$$

for any subset of indices $i_1, \dots, i_k$ with $k=2,\dots,n$.

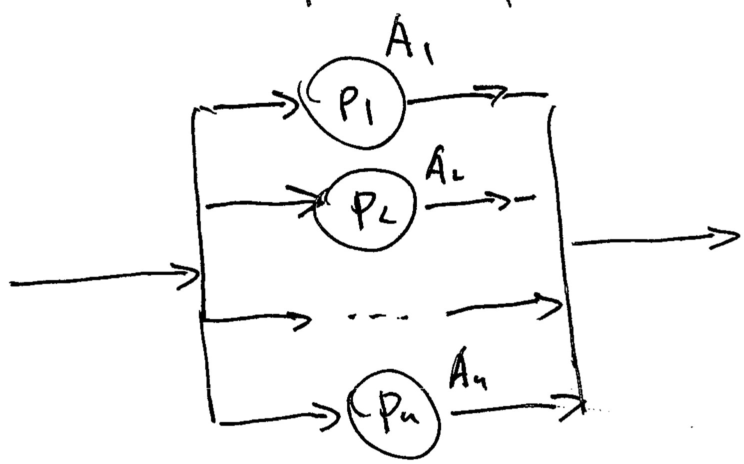
* Systems comprise blocks of independent elements in series or parallel. Here each element works with probability p_i independently of the others.

— block in series



$$P(\text{system working}) = P(A_1 \cap A_2 \cdots \cap A_n)$$
$$= P(A_1)P(A_2)\cdots P(A_n)$$
$$= p_1 \cdot p_2 \cdots p_n$$

— block in parallel



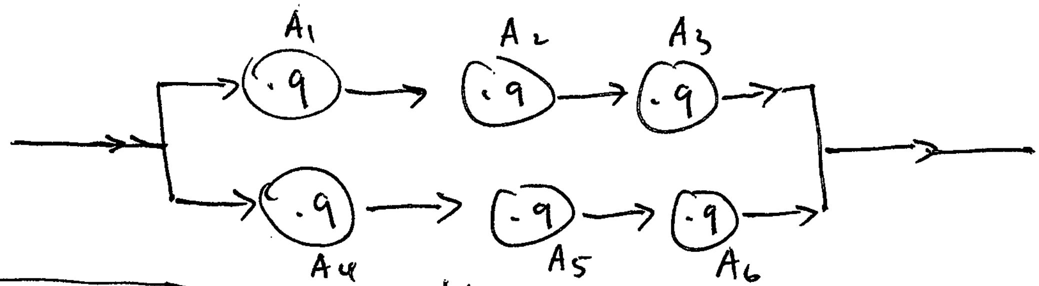
$$P(\text{system working}) = P(A_1 \cup A_2 \cup \cdots \cup A_n)$$
$$= P\left[(A_1^c \cap A_2^c \cap \cdots \cap A_n^c)^c\right]$$
$$= 1 - P(A_1^c \cap A_2^c \cap \cdots \cap A_n^c)$$
$$= 1 - P(A_1^c)P(A_2^c)\cdots P(A_n^c)$$
$$= 1 - (1-p_1)(1-p_2)\cdots(1-p_n)$$

Rmk: The exclusion-inclusion formula

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$$
$$- P(A_1 \cap A_2) - P(A_1 \cap A_3) - \cdots - P(A_{n-1} \cap A_n)$$
$$+ P(A_1 \cap A_2 \cap A_3) + \cdots + P(A_{n-2} \cap A_{n-1} \cap A_n)$$
$$+ (-1)^{n-1} P(A_1 \cap \cdots \cap A_n)$$
$$= \sum_{k=1}^{n} (-1)^k \sum_{1 \le j_1 \le \cdots \le j_k \le n} P(A_{j_1} \cap \cdots \cap A_{j_k})$$
$$= \sum_{k=1}^{n} (-1)^k \sum_{1 \le j_1 \le \cdots \le j_k \le n} p_{j_1} p_{j_2} \cdots p_{j_k} \implies$$

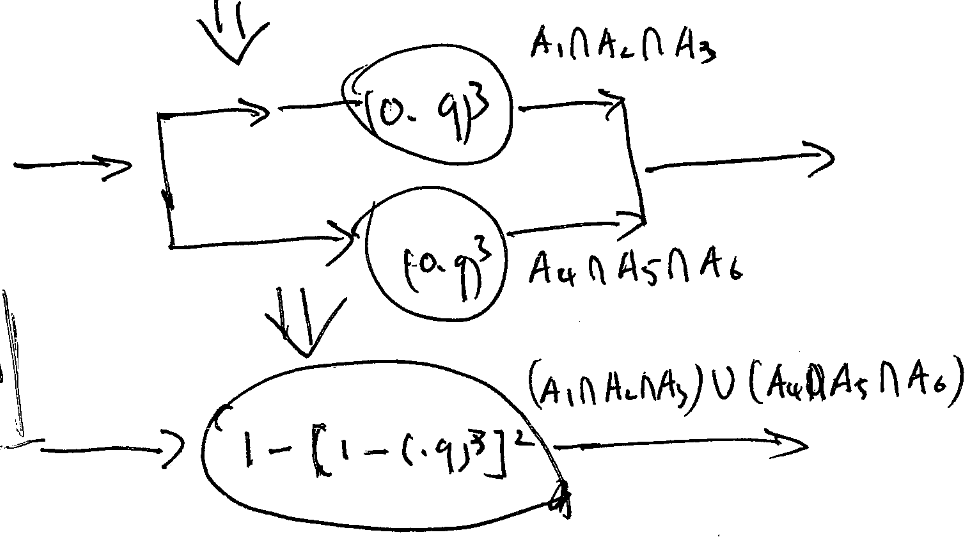
- Example 2.35

  - system is series-parallel



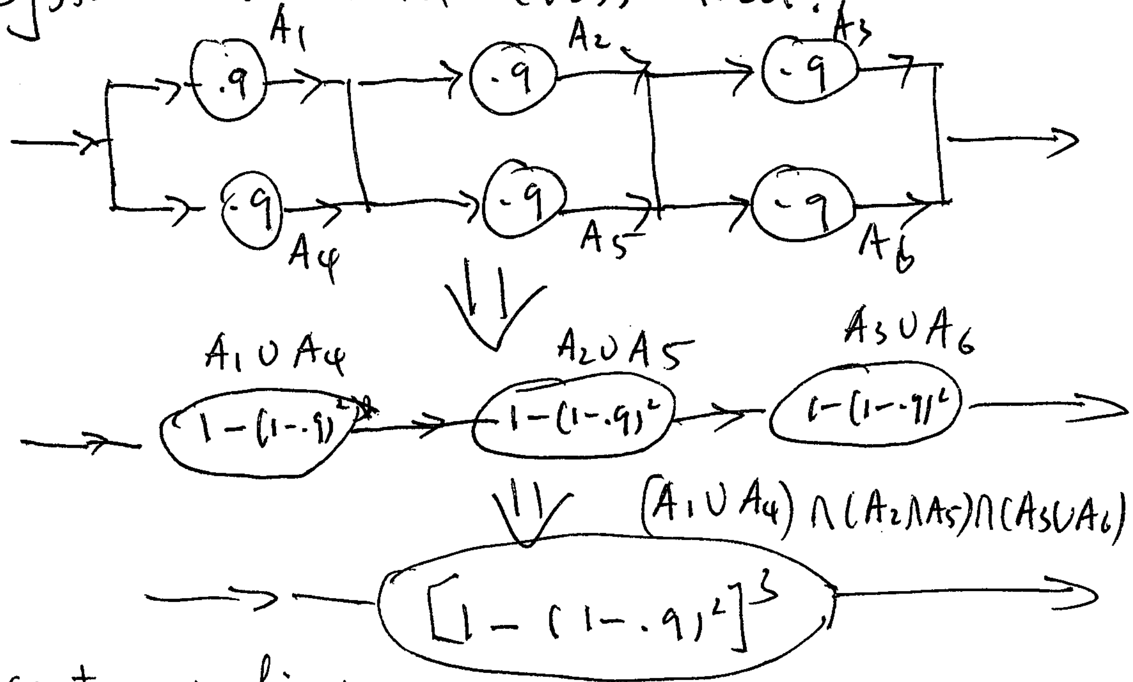
P (system working) can be found by repeatly combining block.

Each block works independently of others



$$P(\text{system working}) = 1 - [1-(.9)^3]^2 = .927$$

  - System in total-cross-tied.



$$P(\text{system working}) = [1-(1-.9)^2]^3 = .970$$