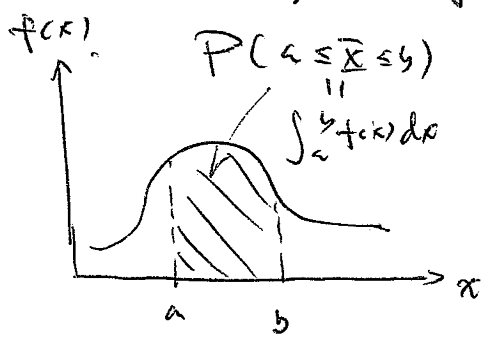
Continuous Random Variables and Probability Distribution

* Definition: The random variable $\underline{X}$ is said to be <u>continuous</u>, with <u>probability density function</u> (pdf) $f(x)$, if for all $a \le b$,

$$P(a \le \underline{X} \le b) = \int_a^b f(x)\,dx$$

* Remarks: 1°/ The graph of $f(x)$ is called the density curve.

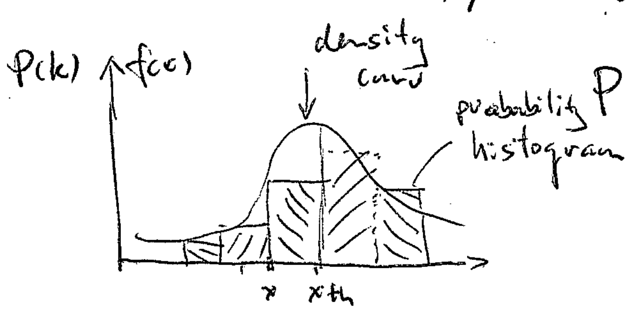


Probability that $\underline{X}$ takes on a value in the interval $[a, b]$

$\|$

Area above this interval and under the graph of the pdf $f(x)$.

2°/ When h is small and $f(x)$ is smooth,

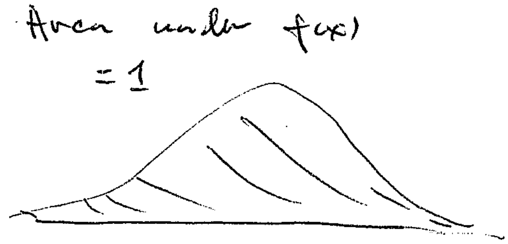


$$P(x \le \underline{X} \le x+h) = \int_x^{x+h} f(x')\,dx'$$

$$\simeq h\,f(x)$$

$\Rightarrow$ discrete probability distribution can be approximated by continuous probability density.

3°. Obviously we have



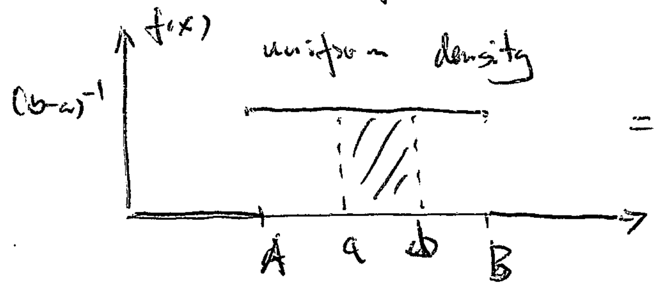
$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

* Example – Uniform (distribution) density

<u>Def.</u> A continuous rv X is said to have a <u>uniform distribution</u> on the interval $[A, B]$ if the pdf of X is

$$f(x; a, b) = \begin{cases} \dfrac{1}{B-A}, & x \in [A, B] \\ 0 & \text{otherwise} \end{cases}$$



= approximation to the discrete uniform distribution on $\{[A], [A]+1, \ldots, [B]\}$.

Then for any $A \leq a \leq b \leq B$,

$$P(a, b) = (b-a)(B-A)^{-1}$$

= depends only on the width $b-a$ of the interval.

* Key rule for densities.

Let X have pdf $f(x)$. Then for $C \subseteq \mathbb{R}$,

$$P(X \in C) = \int_{x \in C} f(x)\,dx$$

* Basic difference between continuous and discrete rvs.

Let X be a continuous rv with pdf $f(x)$.

$$P(X = x) = \int_x^x f(u)\,du = 0, \quad \forall x \in \mathbb{R}$$

and (zero probability condition).

$$P(a \le X \le b) = P(a < X < b) = P(a \le X < b) = P(a < X \le b)$$

If X is discrete, $P(X = x)$ is not zero for any x possible values, and all four of these probabilities are different if both a and b are possible values.

* Example: Exponential density

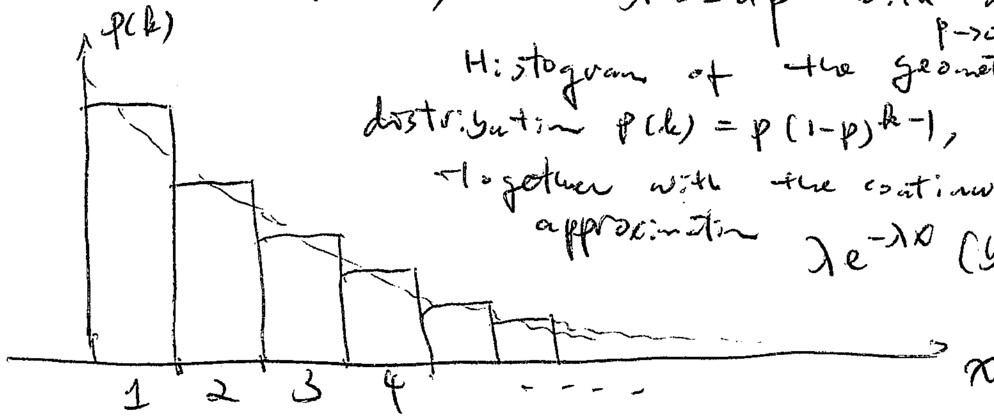
This most important density arose as an approximation to the geometric distribution with

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

= approximation to the geometrical pmf: $p(k) = p(1-p)^{k-1}$

where $\lambda > 0$ is fixed, and $\lambda := np$ with n large, $p \to 0$.

waiting



Histogram of the geometric distribution $p(k) = p(1-p)^{k-1}$, together with the continuous approximation $\lambda e^{-\lambda x}$ (broken line).

Clearly we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

* The cumulative distribution function (cdf)

Definition. Let X have pdf $f(x)$. Then the cdf $F(x)$ of X is given by

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = P(X \le x) = P(X < x).$$

__Remarks__: 1°/ The cdf $F(x)$ is defined in terms of the pdf $f(x)$, via __integration__.

2°/ The pdf $f(x)$ can be derived from the cdf $F(x)$ by __differentiation__:

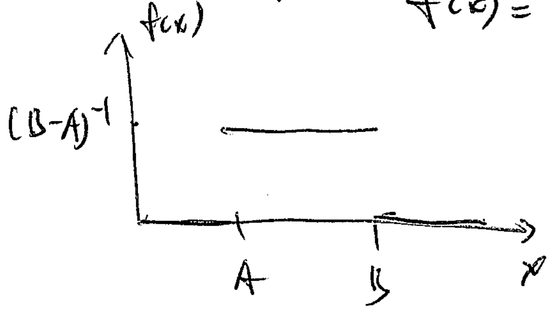
__Proposition__:

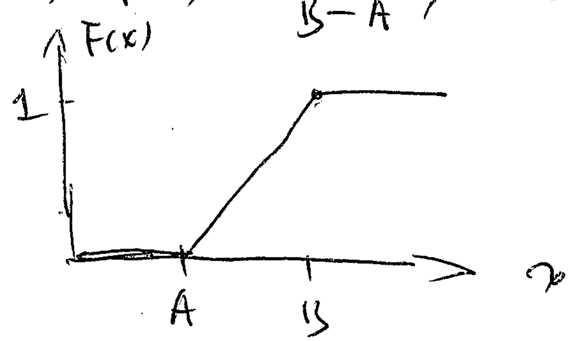
$$\boxed{f(x) = \frac{d}{dx}[F(x)] = F'(x)}$$

(fundamental theorem of calculus)

Examples: 1°/ uniform distribution

$$f(x) = (B-A)^{-1} \Rightarrow F(x) = \frac{x-A}{B-A}, \quad x \in [A,B].$$

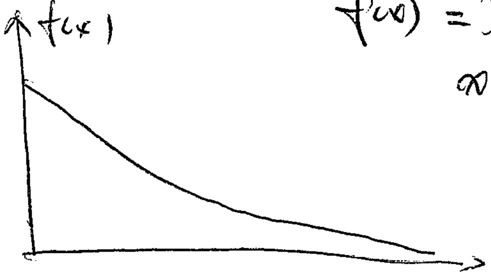


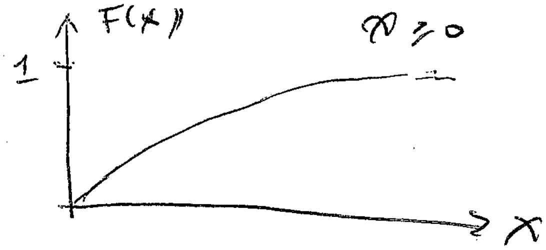


2°/ exponential distribution.

$$f(x) = \lambda e^{-\lambda x} \Rightarrow F(x) = 1 - e^{-\lambda x}$$

$x \geq 0$ $\qquad$ $x \geq 0$





* Using $F(x)$ to compute probability.

$$P(X > a) = 1 - F(a)$$

$$P(a \leq X \leq b) = F(b) - F(a).$$