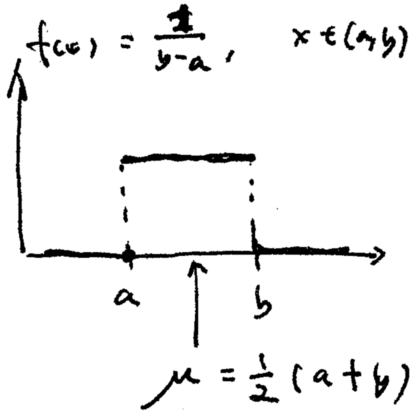
Expectation

Definition. The expected or mean value of a continuous r.v. $\underline{X}$ with pdf $f(x)$ is

$$\mu_{\underline{X}} = E(\underline{X}) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

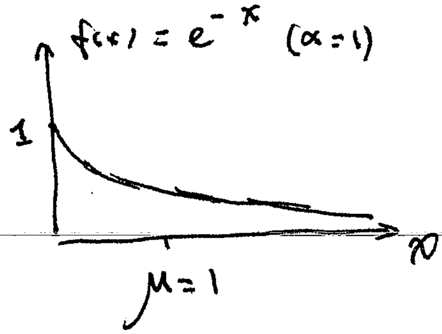
Examples. 1°/ uniform density: Let $\underline{X}$ be uniform on (a, b). Then



$$E(\underline{X}) = \int_a^b x\, f(x)\, dx = \int_a^b \frac{x}{b-a}\, dx$$

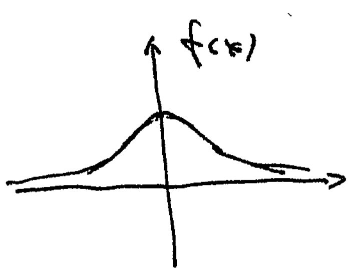
$$= \frac{1}{2}\left(\frac{b^2 - a^2}{b-a}\right) = \frac{1}{2}(a+b)$$

2°/ exponential density. Let $\underline{X}$ be exponential with pdf $f(x) = \alpha e^{-\alpha x}$, $x \geq 0$ $(\alpha > 0)$. Then



$$E(\underline{X}) = \int_0^{\infty} x\, \alpha\, e^{-\alpha x}\, dx = 1/\alpha$$

3°/ normal density: Let $\underline{X}$ be the standard normal r.v. with $f(x) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2)$. Then



$$E(\underline{X}) = \int_{-\infty}^{\infty} x\, f(x) = 0, \text{ by symmetry.}$$

Expectation of functions

Theorem: Let X be a continuous rv with pdf $f(x)$ and $Y = h(X)$. Then

$$\left(E[h(X)] =\right) E[Y] = \mu_Y = \int_{-\infty}^{\infty} h(x)\, f(x)\, dx$$

Rmks: 1°/ The point of Theorem is that we do not need to find the pdf of Y in order to find its mean.

2°/ Linearity of expectation

$$E\left[g(X) + h(X)\right] = E[g(X)] + E[h(X)]$$

3°/ linear transformation

$$E[aX + b] = a\,E[X] + b$$

Variance:

Definition: The variance of a continuous rv X with pdf $f(x)$ and mean value μ is

$$\sigma^2 = V(X) = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

The standard deviation of X is $\sigma = \sqrt{V(X)}$.

Proposition
$$V(\underline{X}) = E(\underline{X}^2) - [E(\underline{X})]^2$$

Proof:
$$V(\underline{X}) = E[(\underline{X}-\mu)^2] = E[\underline{X}^2 - 2\mu\underline{X} + \mu^2]$$

(linearity of expectation)
$$= E(\underline{X}^2) + E[-2\mu\underline{X} + \mu^2]$$

(linear transformation)
$$= E(\underline{X}^2) - 2\mu E[\underline{X}] + \mu^2$$
$$= E(\underline{X}^2) - 2[E(\underline{X})]^2 + [E(\underline{X})]^2$$
$$= E(\underline{X}^2) - [E(\underline{X})]^2$$

Remark: Proposition (linear transformation)
$$V(a\underline{X}+b) = a^2 V(\underline{X})$$

~~Proof~~ Proof:
$$V(a\underline{X}+b) = E[(a\underline{X}+b-\mu_{a\underline{X}+b})^2]$$
$$= E[(a\underline{X}+b-a\mu_{\underline{X}}-b)^2] = E[(a\underline{X}-a\mu_{\underline{X}})^2]$$
$$= E[a^2(\underline{X}-\mu_{\underline{X}})^2] = a^2 E[(\underline{X}-\mu_{\underline{X}})^2]$$
$$= a^2 V(\underline{X}).$$

__Example – normal density:__ Let X be the standard normal rv with $f(x) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2)$. Then $\sim N(0,1)$

$$V(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \left.\frac{x\, e^{-x^2/2}}{(2\pi)^{\frac{1}{2}}}\right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(x)\,dx$$

$$= 1.$$

Now let $Y = \mu + \sigma X$. Then Y has the pdf

$$N(\mu, \sigma^2) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}$$

Thus

$$E[Y] = E[\mu + \sigma X] = \mu$$

$$V(Y) = E[(Y-\mu)^2] = \sigma^2 V(X) = \sigma^2$$

__Moment Generating Function__

__Definition:__ The mgf of a continuous rv X with pdf $f(x)$ is

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

__Theorem:__

$$E(X^r) = M_X^{(r)}(0).$$

Proof. $M_X^{(r)}(t) = \int_{-\infty}^{\infty} x^r e^{tx} f(x)\,dx \Rightarrow E(X^r) = M_X^{(r)}(0)$.