# Transformation of a random variable

General problem: Given random variables $X$ and $Y$, such that

$$Y = g(X)$$

for some function $g$, what is the distribution (pdf) $f_Y$ of $Y$ in terms of that (pdf) $f_X$ $X$?

Rule: In answering this, the cdf appears much more often in dealing with continuous r.v. than it did in discrete case. (remember that $P(X = x) = 0$)

$g(\cdot)$ is a one-to-one function: The answer is straightforward.

Theorem: Let $g(\cdot)$ be a one-to-one function with its inverse $h(\cdot)$. Then

$$f_Y(y) = f_X[h(y)]\,|h'(y)|$$

Proof: ~~Assume that~~ First find the cdf

$$F_Y(y) = P(Y \le y) = P[g(X) \le y] = \underset{\text{increasing}}{P[X \le h(y)]}$$

$$= \begin{cases} F_X[h(y)] & \text{: } (h\text{: increasing}) \\ 1 - F_X[h(y)] & (h\text{: decreasing}) \end{cases}$$

$\underset{\text{decreasing}}{1 - P(X \le h(y))}$

Now differentiate the cdf and letting $x = h(y)$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{d}{dy}\left[ F_X[h(y)] \right] \\ -\frac{d}{dy}[\quad] \end{cases} = \left(\frac{d}{dx} F_X(x)\right) \frac{dx}{dy}$$

$$= f_X(x)\, h'(y) = \begin{cases} f_X(h(y))\, h'(y) \\ -( \quad ) \end{cases} = f_X(h(y))\,|h'(y)|$$

The pdf's must be positive, the absolute value with be needed of the derivative if it is negative.

<u>Example: scaling and shifting</u>: $Y = aX + b$, $a > 0$

$x = \frac{1}{a}(y - b)$, $\frac{dx}{dy} = \frac{1}{a}$

$$f_Y(y) = \frac{1}{a} f\left(\frac{y-b}{a}\right)$$

Let $X \sim N(\mu, \sigma^2)$

$$f_Y(y) = \frac{1}{a} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ - \frac{\left[(y-b)/a - \mu\right]^2}{2\sigma^2} \right\}$$

$$= \frac{1}{\sqrt{2\pi}(\sigma a)} \exp\left\{ - \frac{\left[y - (a\mu + b)\right]^2}{2(\sigma a)^2} \right\}$$

Then $Y \sim N(a\mu + b, a^2\sigma^2)$.

As a special case, take $a = \frac{1}{\sigma}$, $b = -\frac{\mu}{\sigma}$,

then $Y = \frac{X - \mu}{\sigma} \sim N(0, 1)$.

The general case: can be tacked in much the same way. The basic idea is rather obvious.

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

Then, differentiate to get the pdf of $Y$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[P(g(X) \le y)\right].$$

Now, when $g(\cdot)$ is a "friendly function", we should obtain useful expression for $f_Y(y)$.

Example: Normal density

Let $X$ be uniform on $(0,1)$ with

$$f_X(x) = \begin{cases} 1, & x \in (0,1) \\ 0 & \end{cases}$$

Suppose that $Y = -\lambda^{-1} \log X$, $\lambda > 0$.

$$\boxed{\begin{aligned} x &= e^{-\lambda y} \\ f_Y(y) &= f_X(e^{-\lambda y})\lambda e^{-\lambda y} \\ &= \lambda e^{-\lambda y} \end{aligned}}$$

Then

$$F_Y(y) = P(Y \le y) = P(\log X \ge -\lambda y)$$
$$= P(X \ge e^{-\lambda y}) = 1 - e^{-\lambda y}$$
$$f_Y(y) = \frac{d}{dy} F_Y(y) = \lambda = e^{-\lambda y}$$

and $Y$ has an exponential density

cdf of $\mathbb{X}$

<u>Example</u>: Let $Y = g(\mathbb{X}) := F_{\mathbb{X}}(\mathbb{X})$.

uniform density

$F_{\mathbb{X}}$ is increasing

Then, as above

$$F_Y(y) = P(Y \le y) = P(F_{\mathbb{X}}(\mathbb{X}) \le y) = P(\mathbb{X} \le F_X^{-1}(y))$$
$$= F_X(F_{\mathbb{X}}^{-1}(y)) = y \quad \rightarrow \quad f_Y(y) = \begin{cases} 1, & y \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

and $Y$ is uniform on $(0,1)$

<u>Example</u>. Let $f_Y(y)$ (and $F_Y(y)$) be specified. Define $g$ to be the inverse function of $F_Y$,

any density

so $h(y) = F_Y(y)$.

Let $\mathbb{X}$ be uniform on $[0,1]$, and

$$Y = g(\mathbb{X}) = F_Y^{-1}(\mathbb{X}).$$

Then

$$f_Y(y) = (1)\ \frac{d}{dy} F_Y(y)$$

$\Rightarrow$ build any rv with specific cdf from uniform rv.

Example power.

$$Y = X^2.$$

Note that the function is not one-to-one.

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

So that $Y$ has the pdf.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2\sqrt{y}} \{ f_X(\sqrt{y}) + f_X(-\sqrt{y}) \} \qquad \frac{d}{dy}\sqrt{y} = \frac{1}{2\sqrt{y}}$$

Now, Let $X \sim N(0,1)$. Then

$$f_X(\sqrt{y}) + f_X(-\sqrt{y})$$

$$= 2 f_X(\sqrt{y}) = \frac{2}{\sqrt{2\pi}} \exp\left\{ -\frac{(\sqrt{y})^2}{2} \right\}$$

Then

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} \exp -\frac{y}{2}, \quad y > 0$$

and $Y$ has the chi-squared distribution with $(\nu = 1)$

i.e. 1 degree of freedom.