Jointly distributed random variables

* Joint distributions for two discrete r.v's

Definition: Let X and Y be two discrete r.v's defined on the sample space Ω of an experiment. The joint pmf $p(x,y)$ is defined by

$$p(x,y) = P(X = x \text{ and } Y = y), \quad \forall (x,y) \in \mathbb{R}^2.$$

Remarks: 1°/ The pmf $p(x,y)$ satisfies

$$0 \le p(x,y) \le 1, \quad \forall (x,y) \in \mathbb{R}^2$$

$$\sum_{x,y} p(x,y) = \sum_x \sum_y p(x,y) = 1$$

2°/ Let $A \subseteq \mathbb{R}^2$. Then

$$P([X,Y] \in A) = \sum_{(x,y) \in A} p(x,y)$$

Examples: 1°/ Pair of dice. Two fair dice are rolled, yielding the scores X and Y. The joint pmf is

$$p(x,y) = \frac{1}{36}, \quad 1 \le x, y \le 6$$

$p(x,y)$	1	2	3	4	5	6
1			⋅			
2			⋮			
3	⋅	⋅	$\frac{1}{36}$	-		-
4						
5			⋮			
6			⋮			

2°/ <u>Flipping a coin.</u> Suppose a coin shows a head with probability p, or a tail with probability $q = 1-p$. Let X be the number of flips until the first head, and Y the number of flips until first tail. Then the joint pmf is

You flip the coin repeatedly

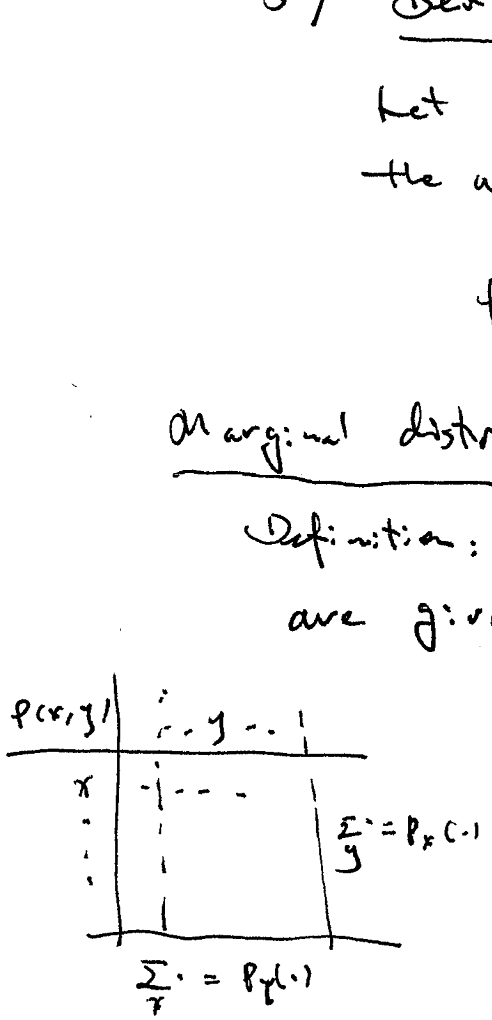
$$p(1, y) = p^{y-1} q, \quad y \geq 2$$
$$p(x, 1) = p\, q^{x-1}, \quad x \geq 2$$

3°/ <u>Bernoulli trials</u>; In n Bernoulli trials let X be the number of successes and Y the number of failures. Then $x+y=n$, and

$$p(x,y) = \binom{n}{x} p^x (1-p)^y .$$

<u>Marginal distributions.</u>

Definition: The marginal pmf's of X and Y are given by



$$p_X(x) = \sum_y p(x,y)$$
$$p_Y(y) = \sum_x p(x,y)$$

Examples: e/ revised dice. The joint pmf $p(x,y)=\frac{1}{36}$.

The marginal pmf's:
$$\begin{cases} p_X(x) = \sum_{y=1}^{6} p(x,y) = \frac{1}{6} \\ p_Y(y) = \sum_{x=1}^{6} p(x,y) = \frac{1}{6} \end{cases}$$

Remark: The joint pmf $p(x,y)$ always yields the marginal $p_X(x)$ and $p_Y(y)$. However, the converse is not true: the marginals do not determine the joint distribution.

* Joint density for two continuous rv's

Definition: Let X and Y be continuous rv's.

Then $f(x,y)$ is the joint pdf for X and Y if $\forall A \subset \mathbb{R}^2$,

$$P[(X,Y)\in A] = \iint_A f(x,y)\,dx\,dy$$

In particular, if $A=[a,b]\times[c,d]$

$$P(a\le X\le b,\ c\le Y\le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Remark: The pdf $f(x,y)$ satisfies

$$f(x,y)\ge 0$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

Example Let X and Y have joint pdf

$$f(x,y) = cxy, \quad 0 \le x,y \le 1$$

What is c?

Solution: We know $f \ge 0$, so $c \ge 0$

And $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^1 cxy\,dy\,dx$$

$$= c\int_0^1 x\,dx \int_0^1 y\,dy = \frac{c}{4} = 1$$

Hence $c = 4$.

Marginal density: The marginal pdfs of X and Y are

$$\begin{cases} f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \\ f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \end{cases}$$

Examples: 1°/ Let X and Y have the joint pdf

$$f(x,y) = 4xy, \quad 0 < x,y \le 1$$

Find the marginal density $f_X(x)$, $f_Y(y)$

Solution:

$$\begin{cases} f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 4xy\,dy \\ \quad = 2x \\ f_Y(y) = 2y \end{cases}$$

2. Let X and Y have the joint pdf

$$f(x,y) = cxy, \qquad 0 \le x, y \le 1, \quad x+y \le 1$$

Find (i) the value of c, (ii) $P(X+Y \le 0.5)$ (iii) the marginal pdf $f_X(x)$

Solution: (i)

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^{1-x} cxy\,dy\,dx$$

$$= \int_0^1 cx\left[\frac{y^2}{2}\Big|_0^{1-x}\right]dx = \int_0^1 \frac{c}{2}x(1-x)^2\,dx$$

$$= \frac{c}{24}$$

Then $c = 24$

(ii) $P(X+Y \le 0.5)$

(x,y):
$A = \{x+y \le 0.5\}$

$$= \iint_A f(x,y)\,dy\,dx = \int_0^{0.5}\int_0^{0.5-x} 24xy\,dy\,dx$$

$$= 0.0625$$

(iii)

$0 < x \le 1$, $\quad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 24xy\,dy = 12x(1-x)^2$

* Independent rv's.

<u>Definition</u>: Two rv's X and Y are independent if for every pair of x and y values

$p(x,y) = p_X(x) \cdot p_Y(y)$ when X and Y are discrete

or

$f(x,y) = f_X(x) \cdot f_Y(y)$ when X and Y are continuous.

<u>Remark</u>: if X and Y are independent, then for any $\{X \in A\}$ and $\{Y \in B\}$,

$P(X \in A,\ Y \in B) = P(X \in A)\, P(X \in B)$

in particular,

$P(a \le X \le b,\ c \le Y \le d) = P(a \le X \le b) \cdot P(c \le Y \le d)$.

<u>Examples</u>: w/ independent normal rv's. Let X and Y be independent standard normal rv's.

Then, their joint pdf is

$f(x,y) = f_X(x)\, f_Y(y) = \frac{1}{2\pi} \exp\left\{-\frac{x^2+y^2}{2}\right\}$.