Math 3338: Probability (Fall 2006)

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Chapter Two: Probability (I)

2.1 Sample Spaces and Events

The sample space of an experiment

- **Experiment:** any action or process whose outcome is subject to uncertainty.
- **Sample space of an experiment:** denoted by Ω, the set of all possible outcomes of that experiment.
- **Sample point:** denoted by ω, a point in Ω.
  - The sample space provides a model of an ideal experiment in the sense that, by definition, every thinkable outcome of the experiment is completely described by one, and only one, sample point.

$$\forall \text{outcome of the experiment}, \exists! \omega \in \Omega \Rightarrow \text{outcome} = \omega$$

- **Examples:**
  - **2.1 :** For an experiment consisting of examining a single fuse to see whether it is defective, the sample space $\Omega_1 = \{N, D\}$, where N represents not defective, D represents defective.
  - **2.2 :** For an experiment consisting of examining three fuses in sequence, the sample space $\Omega_3 = \Omega_1 \times \Omega_1 \times \Omega_1 = \{(\omega_1, \omega_2, \omega_3), \omega_i \in \{N, D\}\}$.
  - **2.4 :** For an experiment consisting of testing each battery as it comes off an assembly line until we first observe a success, $\Omega = \{S, FS, FFS, FFFS, \ldots\}$, which contains an infinite number of possible outcomes.

Events

- **Event:** denoted by capital letters, is a subset of Ω.
  - It is meaningful to talk about an event A only when it is clear for every outcome of the experiment whether the event A has or has not occurred.

$$\forall \omega \in \Omega, \text{ we have } \omega \in A \text{ or } \omega \notin A.$$

- **Elementary event:** also called **simple event**, is an event consisting of exactly one outcome: elementary event $=$ sample point.

An event A occurs if and only if one of the elementary event ω in A occurs.

- **Certain event :** $= \Omega$, which always occurs regardless the outcome of the experiment.
- **Impossible event :** $= \emptyset$.
- **Example 2.7:**
  - For Ω defined in example 2.4, compound events include

$$\begin{aligned} A &= \{S, FS, FFS\} \\ &= \text{the event tat at most three batteries are examined} \\ E &= \{FS, FFFS, FFFFFS, \ldots\} \\ &= \text{the event that an even number of batteries are examined.} \end{aligned}$$

Operations with events

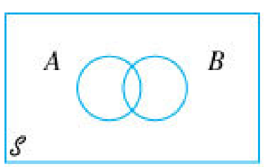
- **Complement:** The complement of an event A is denoted by A^c or A', and is the event that A does not occur. $$A^c = \{\omega | \omega \notin A\}; \quad \Omega^c = \emptyset; \emptyset^c = \Omega, (A^c)^c = A.$$

- **Union:** The union $A \cup B$ of two events A and B is the event consisting of the occurrence of at least one of the events A and B.
$$A \cup B = \{\omega | \omega \in A \text{ or } \omega \in B\}.$$

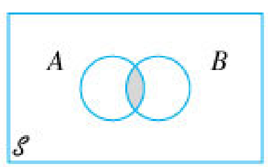
- **Intersection:** The intersection $A \cap B$ of two events A and B is the event consisting of the occurrence of both events A and B.
$$A \cap B = \{\omega | \omega \in A \text{ and } \omega \in B\}.$$

- **Mutually exclusive events:** Two events are said mutually exclusive if $A \cap B = \emptyset$, i.e., the event $A \cap B$ is impossible.

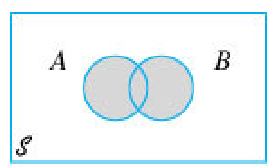
- **Figure 2.1 - Venn diagrams :** An even is nothing but a set, so *elementary set theory* can be used to study events.



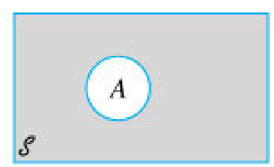
(a) Venn diagram of events A and B



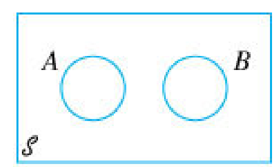
(b) Shaded region is $A \cap B$



(c) Shaded region is $A \cup B$



(d) Shaded region is A'



(e) Mutually exclusive events

Relations among events

- $\mathbf{A} \subset \mathbf{B}$: The occurrence of A implies the occurrence of B, i.e., if $\omega \in A$, then $\omega \in B$.
- $\mathbf{A} \supset \mathbf{B}$: The occurrence of A is implied by the occurrence of B, i.e., if $\omega \in B$, then $\omega \in A$.
- $\mathbf{A} = \mathbf{B}$: The events A and B are identical.

$$A = B \text{ if and only if } A \subset B \text{ and } B \subset A.$$

- $\mathbf{A} \cap \mathbf{B^c}$: A but not B occurs.
- $\mathbf{A} - \mathbf{B}$: If $A \supset B$, we denote $A \cap B^c$ by $\mathbf{A} - \mathbf{B}$.
- $\mathbf{A} + \mathbf{B}$: If $A \cap B = \emptyset$, we denote $A \cup B$ by $\mathbf{A} + \mathbf{B}$.
- $\mathbf{A} \cap \mathbf{B} = \emptyset$: means that $A \subset B^c$ and $B \subset A^c$, i.e., if A and B are mutually exclusive, the occurrence of A implies the non-occurrence of B and vice versa.
- $\mathbf{A} - \mathbf{A} \cap \mathbf{B}$: the occurrence of A but not of both A and B. Thus $A - A \cap B = A \cap B^c$.
- **Indicator:** The indicator of an event A is $I_A : \Omega \mapsto \{1, 0\}$ s.t. $I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$

  We have then

$$A = B \text{ if and only if } I_A = I_B$$

$$I_{A \cap B} = I_A \cdot I_B$$

$$I_{A \cup B} = I_A + I_B - I_A \cdot I_B$$

Various Laws

- **Commutative law:**
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
- **Associative law:**
$$(A \cup B) \cup C = A \cup (B \cup C) =: A \cup B \cup C$$
$$(A \cap B) \cap C = A \cap (B \cap C) =: A \cap B \cap C$$
- **Distributive law:**
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- **De Morgan's law:**
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$
- **Proofs :** Based on the relation
$$\text{Left} = \text{Right} \quad \text{if and only if} \quad \begin{cases} (1) \quad \text{Left} \subset \text{Right} \\ (2) \quad \text{Right} \subset \text{Left} \end{cases}$$