Math 3338: Probability (Fall 2006)

Jiwen He

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http://math.uh.edu/~jiwenhe/math3338fall06.html

2.2 Axioms, Interpretations, and Properties of Probability

Definition and three axioms

- **Definition:** A *probability measure* on the sample space Ω is a function of *events*,

$$P : A \subset \Omega \mapsto P(A) \in R$$

satisfying three axioms

Axiom 1: For any event A, $P(A) \geq 0$.

Axiom 2: $P(\Omega) = 1$.

Axiom 3: If $A_1, A_2, \ldots$ is an infinite collection of *disjoint events*, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i).$$

- **Proposition:**
  1. $P(\emptyset) = 0$ where $\emptyset$ is the null event.
  2. Axiom 3 is valid for a *finite* collection of events.

Proof: 1. Consider $A_1 = \emptyset, A_2 = \emptyset, \ldots$. Axiom 3 gives $P(\emptyset) = \sum P(\emptyset)$, which can happen only if $P(\emptyset) = 0$. 2. Consider k disjoint events $A_1, A_2, \ldots, A_k$ and append to these the infinite collection $A_{k+1} = \emptyset, A_{k+2} = \emptyset, \ldots$. Again invoking Axiom 3,

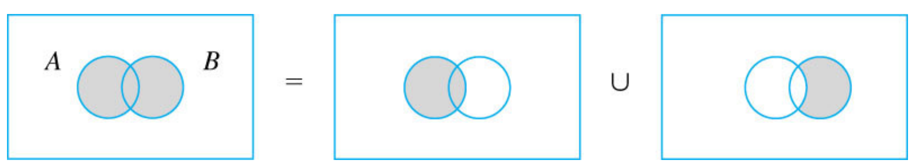
$$P(\bigcup_{i=1}^{k} A_i) = P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{k} P(A_i)$$

as desired.

Deductions from the axioms

- **Proposition:**
  1. For any event A, $P(A) = 1 - P(A^c)$. Then $0 \leq P(A) \leq 1$.
  2. For any two events A and B,
  $$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
  3. For any three events A, B, and C,
  $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$
  4. For any n events $A_1, A_2, \ldots, A_n$,
  $$P\left(\bigcup_{i=1}^{n} A_i\right) = P_1 - P_2 + P_3 - P_4 + \cdots + (-1)^{n+1} P_n$$
  where
  $$P_1 = \sum_{i=1}^{n} P(A_i), P_2 = \sum_{1 \leq i \leq j \leq n} P(A_i \cap A_j), P_3 = \sum_{1 \leq i \leq j \leq k \leq n} P(A_i \cap A_j \cap A_k), \ldots$$

Proof: 1.– 3. by the partition, 4. by induction.



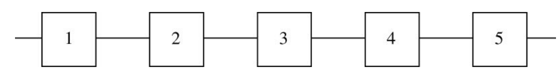
Examples

- **2.11:** $\Omega = \{U, D\}$. For any $0 \le p \le 1$, $P(U) = p$, $P(D) = 1 - p$ is a probability assignment consistent with the axioms: $1 = P(\Omega) = P(U) + p(D)$.
- **2.11:** $\Omega = \{S, FS, FFS, FFFS, \cdots$. For any $0 \le p \le 1$, suppose the probability of any particular battery being satisfactory is p. Then it can be shown that $P(S) = p$, $P(FS) = (1-p)p,$, $P(FFS) = (1-p)^2 p, \ldots$, is a probability assignment consistent with the axioms:

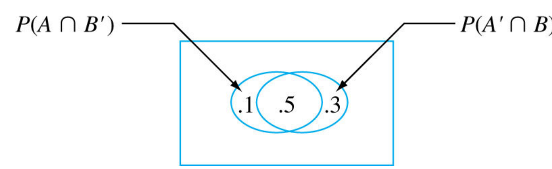
$$1 = P(\Omega) = P(S) + p(FS) + P(FFS) + \cdots = p \sum_{i=0}^{\infty} (1-p)^i = p \frac{1}{1-(1-p)}.$$

- **2.13:** Let A be the event that the system fails. Suppose the probability of any particular component being satisfactory is p, $0 \le p \le 1$.

$$P(A) = 1 - P(A^c) = 1 - P(SSSSS) = 1 - p^5.$$



- **2.14:** P(subscribes to at least one of the two newspapers)



$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .8 - .5 = .9$$

More properties

- **Proposition:**

  1. For any two events A and B such that $A \subset B$,

  $$P(B - A) = P(B) - P(A).$$

  Then

  $$P(A) \leq P(B)$$

  2. (Boole's Inequality) For any n events $A_1, A_2, \ldots, A_n$,

  $$P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$$

Proof:

1. By the partition: $B = A + (B - A)$, we have $P(B) = P(A) + P(B - A)$.
2. By induction. For $n = 2$, it follows that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B).$$

Suppose that it holds for any $n - 1$ events. Then

$$\begin{aligned} P(\bigcup_{i=1}^{n} A_i) &= P((\bigcup_{i=1}^{n-1} A_i) \cup A_n) \leq P(\bigcup_{i=1}^{n-1} A_i) + P(A_n) \\ &\leq \sum_{i=1}^{n-1} P(A_i) + P(A_n) = \sum_{i=1}^{n} P(A_i) \end{aligned}$$

Determining probability systematically

- A simple way to determine probability measure for any *countable* sample space $\Omega = \{E_1, E_2, \ldots, E_i, \ldots\}$ is to
  1. Determine probabilities $P(E_i)$ for all simple events such that
  $$P(E_i) \geq 0, \quad \sum_i P(E_i) = 1.$$
  2. For any event A, the probability $P(A)$ is computed by adding together the $P(E_i)$'s for all E_i's in A:
  $$P(A) = \sum_{E_i \in A} P(E_i)$$

  Thus, $P(\cdot)$ is a function defined for all events and it can be shown that $P(\cdot)$ is a probability measure satisfying the three axioms.

- **Example 2.15 - tossing a die:** $\Omega = \{1, 2, \ldots, 6\} =: \{E_1, E_2, \ldots, E_6\}$. An probability assignment to simple events is
  $$P(E_1) = P(E_3) = P(E_5) = 1/9, \quad P(E_2) = P(E_4) = P(E_6) = 2/9.$$
  Then for the event $A = \{\text{outcome is even}\}$, we have
  $$A = \{E_2, E_4, E_6\}, \quad P(A) = P(E_2) + P(E_4) + P(E_6) = 6/9.$$

Equally likely outcomes

- In the very special case that Ω is finite and contains exactly N point, i.e., $\Omega = \{E_1, E_2, \ldots, E_i, \ldots, E_N\}$ and $|\Omega| = N$, we may attach equal probability p to all N points, $P(E_i) = p$ for every i,

$$1 = \sum_{i=1}^{N} P(E_i) = pN \quad \text{so that} \quad p = \frac{1}{N}.$$

  That is, if there are N possible outcomes, then the probability assigned to each is $1/N$.

- For any event A, with $|A|$ denoting the number of points contained in A. Then

$$P(A) = \sum_{E_i \in A} P(E_i) = \frac{|A|}{|\Omega|}.$$

- Once we have *counted* the number N of outcomes in the sample spaces, to compute the probability of any event we must *count* the number of outcomes contained in that event and take the ratio. **When outcomes are *equally likely*, computing probability reduces to counting.**

- **Example 2.16 - Rolling two dice separately:** $N = 36$. If both the dices are fair, all 36 outcomes are equally likely, so $P(E_i) = 1/36$. Then the event $A = \{\text{sum of two numbers} = 7\}$ consists of the six outcomes: $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$, and $(6, 1)$, so

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6}{36} = 1/6.$$