Math 3338: Probability (Fall 2006)

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Section Number: 10853

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2.3 Counting Techniques (III) - Partitions

Partitioned into k subpopulations

- **Theorem:** Let $r_1, \ldots, r_k$ be integer such that

$$r_1 + r_2 + \cdots + r_k = n, \quad r_i \geq 0.$$

  The number of ways in which a population of n elements can be divided into k ordered parts (partitioned into k subpopulations) of which the first contains r_1 elements, the second r_2 elements, etc. is

$$\frac{n!}{r_1! r_2! \cdots r_k!}.$$

  (The above number is called *multinomial coefficients*.)

- **Proof:** In order to effect the desired partition, we have first to select r_1 elements out of the given n; of the remaining $n - r_1$ elements we select a second group of size r_2, etc. After forming the $(k-1)$st group there remain

$$n - r_1 - r_2 - \cdots - r_{k-1} = r_k$$

  elements, and these form the last group. We conclude that

$$C_{r_1}^{n} C_{r_2}^{n-r_1} C_{r_3}^{n-r_1-r_2} \ldots C_{r_{k-1}}^{n-r_1-\cdots-r_{k-2}} = \frac{n!}{r_1! r_2! \cdots r_k!}$$

  represents the number of ways in which the operation can be performed.

Examples

- **Permutation of n balls that are distinguishable by groups:** Suppose that there are r_1 balls of color no. 1, r_2 balls of color no. 2, ..., r_k balls of color no. k. Their colors are distinguishable, but balls of the same color are not. Of course, $r_1 + \cdots + r_k = n$. The total number of distinguishable arrangements of these n balls are
$$\frac{n!}{r_1!r_2!\cdots r_k!}.$$
- **Dice:** A throw of twelve dice can result in 6^{12} different outcomes, to all of which we attribute equal probabilities. The event that each face appears twice can occur in as many ways as twelve dice can be arrange in six groups of two each $2 + 2 + \cdots + 2 = 12$. Hence the probability of the event is
$$\frac{\frac{12!}{(2!)^6}}{6^{12}} = 0.003438\ldots.$$
- **Bridge:** At a bridge table the 52 cards are partitioned into four equal groups and therefore the number of different situations is $\frac{52!}{(13!)^4} = (5.36\ldots)\cdot 10^{28}$. Let us now calculate the probability that each players has an ace. The four aces can be ordered in $4! = 24$ ways, and each order represents one possibility of giving one ace to each player. The remaining 48 cards can be distributed in $\frac{48!}{(12!)^4}$ ways. Hence the required probability is
$$\frac{24 \cdot 48! \cdot 13^4}{52!} = 0.105\ldots$$

Occupancy problems

- **Placing randomly r balls into n cells:** There are n^r possible distributions. The most important properties of a particular distribution are expressed by its *occupancy numbers* $r_1, \ldots, r_n$ where r_i is the number of balls in the ith cell. Here

$$r_1 + r_2 + \cdots + r_n = r, \quad r_i \geq 0. \tag{1}$$

  We treat the balls as indistinguishable. The distribution of balls is then completely described by its occupancy numbers.

- **Theorem:** The number of distinguishable distributions (i.e. the number of different solution of equation (1)) is

$$A_{r,n} = C_r^{n+r-1} = C_{n-1}^{n+r-1}.$$

  The number of distinguishable distributions in which no cell remains empty is C_{n-1}^{r-1}.

- **Proof:** We use the artifice of representing the n cells by the space between $n+1$ bars and the balls by stars. Thus

$$| \star \star \star | \star |||| \star \star \star \star |$$

  is used as a symbol for a distribution of $r = 8$ balls in $n = 6$ cells with occupancy numbers 3, 1, 0, 0, 0, 4. Such a symbol necessarily starts and ends with a bar, but the remaining $n-1$ bars and r stars can appear in an arbitrary order. In this way it becomes apparent that the number of distinguishable distributions equals the number of ways of selecting r places out of $n+r-1$. The condition that no cell be empty imposes the restriction that no two bars be adjacent. The r stars leave $r-1$ spaces of which $n-1$ are to be occupied by bars: thus we have C_{n-1}^{r-1} choices.

Examples

- **Sampling with replacement and without ordering:** Choose r elements from a population of n elements, with replacement and without ordering.
- **Dice:** There are C_5^{r+5} distinguishable results of a throw with r indistinguishable dice.
- **Partial derivatives:** The partial derivatives of order r of an analytic function $f(x_1, \ldots, x_n)$ of n variables do not depend on the order of differentiation but only on the number of times that each variable appears. Thus each variable corresponds to a cell, and hence there exist C_{n-1}^{n+r-1} different partial derivatives of rth order.

Placing r Balls into n Cells

- Placing r balls into n cells is one way of partitioning the population of r balls. There exist $r!/(r_1! \cdot r_2! \cdots r_n!)$ *distributions with given occupancy numbers* $r_1, r_2, \ldots, r_n$. This formula still involves the order in which the occupancy numbers, or cells, appear, but frequently this order is immaterial. The following example illustrates an exceedingly simple and routine method of solving many elementary combinatorial problems.

- **Configurations of $r = 7$ balls in $n = 7$ cells:** Consider the distributions with occupancy numbers 2, 2, 1, 1, 1, 0, 0 *appearing in an arbitrary order*. These seven occupancy numbers induce a partition of the *seven cells* into three subpopulations consisting, respectively, of the two doubly occupied, the three simply occupied, and the two empty cells. Such a partition into three groups of size 2, 3, and 2 can be effected in

$$\frac{7!}{2! \cdot 3! \cdot 2!}$$

ways. To each particular assignment of our occupancy numbers to the seven cells there correspond

$$\frac{7!}{2! \cdot 2! \cdot 1! \cdot 1! \cdot 0! \cdot 0!} = \frac{7!}{2! \cdot 2!}$$

different distributions of the $r = 7$ balls into the seven cells. Accordingly, *the total number of distributions* such that the occupancy numbers coincide with 2, 2, 1, 1, 1, 0, 0 in some order is

$$\frac{7!}{2! \cdot 3! \cdot 2!} \times \frac{7!}{2! \cdot 2!}.$$