Math 3338: Probability (Fall 2006)

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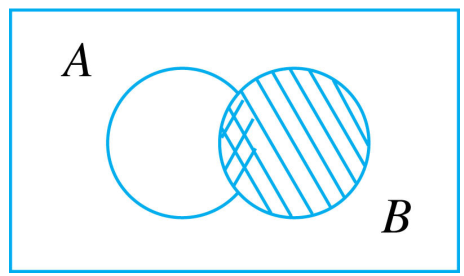
Section Number: 10853

`http://math.uh.edu/~jiwenhe/math3338fall06.html`

2.4 Conditional Probability

Definition of conditional probability

- **Definition:** For any two events A and B with $P(B) > 0$, the *conditional probability of A given B has occurred* is defined by
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
That is, the conditional probability is expressed as a ratio of unconditional probabilities.
- Given that B has occurred, the relevant sample space is no longer Ω but consists of outcomes in B; A has occurred if and only if one of the outcomes in the intersection occurred, so the conditional probability of A given B is proportional to $P(A \cap B)$. The proportionality constant $1/P(B)$ is used to ensure that the probability $P(B|B)$ of the sample space B equals 1.



Examples

- **2.24:** Let $A = \{\text{a line } A \text{ component is selected}\}$, $B = \{\text{the chosen component is defective}\}$.

Line	B	B'
A	2	6
A'	1	9

$$P(A) = \frac{|A|}{|\Omega|} = \frac{8}{18} = .44$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{18}}{\frac{3}{18}} = \frac{2}{3}$$

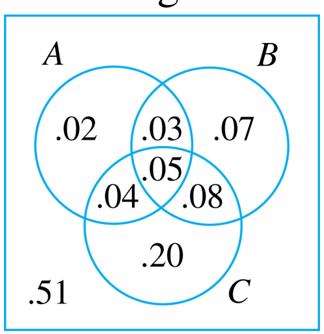
- **2.25:** Let $A = \{\text{memory card purchased}\}$ and $B = \{\text{battery purchased}\}$. Then

$$P(A) = .6, \quad P(B) = .4, \quad P(A \cap B) = .3$$

The conditional probabilities are

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.3}{.4} = .75, \quad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{.3}{.6} = .5.$$

- **2.26:** Reading habits with respect to "Art" (A), "Book" (B), and "Cinema" (C):



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.08}{.23} = .348$$

$$P(A|B \cup C) = \frac{P(A \cap (B \cup C))}{P(B \cup C)} = \frac{.04 + .05 + .03}{.47} = .255$$

$$P(A|A \cup B \cup C) = \frac{P(A \cap (A \cup B \cup C))}{P(A \cup B \cup C)} = \frac{P(A))}{P(A \cup B \cup C)} = \frac{.14}{.49} = .286$$

$$P(A \cup B|C) = \frac{P((A \cup B) \cap C)}{P(C)} = \frac{.04 + .05 + .08}{.37} = .459$$

Multiplication Rule for $P(A \cap B)$

- **Multiplication rule:**

$$\begin{aligned} P(A \cap B) &= P(A|B) \cdot P(B) \\ &= P(B|A) \cdot P(A). \end{aligned}$$

- **2.27:** Four individuals are selected in random order for typing. Only type O+ is desired and only one of the four actually has this type. What is the probability that at least three individuals must be typed to obtain the desired type? Let

$$\Omega = \{\text{four blood types, in which only one type is O+}\}$$
$$B = \{\text{first type is not O+}\}$$
$$A = \{\text{second type is not O+}\}$$

As three of the four types are not O+, and given that the first type is not O+, two of the three left types are not O+, we have

$$P(B) = \frac{3}{4}, \quad P(A|B) = \frac{2}{3}$$

The intersection of A and B gives the event that at least three individuals are typed. The multiplication rule now gives

$$P(A \cap B) = P(A|B) \cdot P(B) = \frac{2}{3} \cdot \frac{3}{4} = .5$$

Multiplication Rule for $P(A \cap B \cap C)$

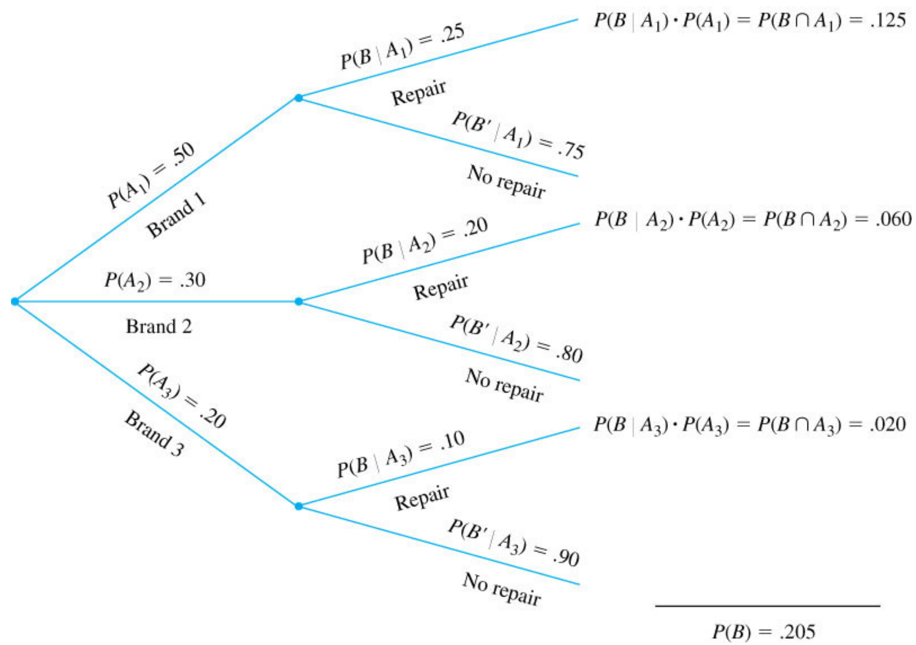
- The multiplication rule is most useful when the experiment consists of several stages in succession, where A_1 occurs first, followed by A_2, and finally A_3: The multiplication

$$P(A_3 \cap A_2 \cap A_3) = P(A_3|A_1 \cup A_2) \cdot P(A_2|A_1) \cdot P(A_1).$$

- **2.28 (2.27 cont.):**

$$\begin{aligned} P(\text{third type is O+}) &= P(\text{third is}|\text{first isn't} \cap \text{second isn't}) \cdot P(\text{second isn't}|\text{first isn't}) \cdot P(\text{first isn't}) \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4} = .25 \end{aligned}$$

- **2.29 - Bayes' theorem:**



The Law of Total Probability

- **Mutually exclusive:** Events $A_1, \ldots, A_k$ are *mutually exclusive* if no two have any common outcomes, so that $A_i \cap A_j = \emptyset$ for any $i, j = 1, \ldots, k$.
- **Exhaustive:** Events $A_1, \ldots, A_k$ are *exhaustive* if one A_i must occur, so that $A_1 \cup \cdots \cup A_k = \Omega$.
- **Theorem:** Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events, so that $A_1 + \cdots + A_k = \Omega$. Then for any other event B,

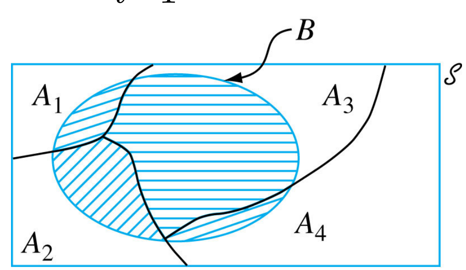
$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

- **Proof:** From the partition of B

$$B = B \cap \Omega = B \cap (A_1 + \cdots + A_k) = B \cap A_1 + \cdots + B \cap A_k = \sum_{i=1}^{k} B \cap A_i$$

it follows that

$$P(B) = P(\sum_{i=1}^{k} B \cap A_i) = \sum_{i=1}^{k} P(B \cap A_i) = \sum_{i=1}^{k} P(B|A_i) \cdot P(A_i).$$



Bayes' Theorem

- **Theorem:** Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events with $P(A_i) > 0$ for $i = 1, \ldots, k$. Then for any other event B for which $P(B) > 0$, we have, for $j = 1, \ldots, k$,

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i) \cdot P(A_i)}{\sum_{j=1}^{k} P(B|A_j) \cdot P(A_j)}.$$

- **Proof:** combine the total probability law and the multiplication rule.
- **Prior and posterior Probabilities:** The computation, provided by Bayes' theorem, of a posterior probability $P(A_i|B)$ from given prior probabilities $P(A_i)$ and conditional probabilities $P(B|A_i)$, which occupies a central position in elementary probability.

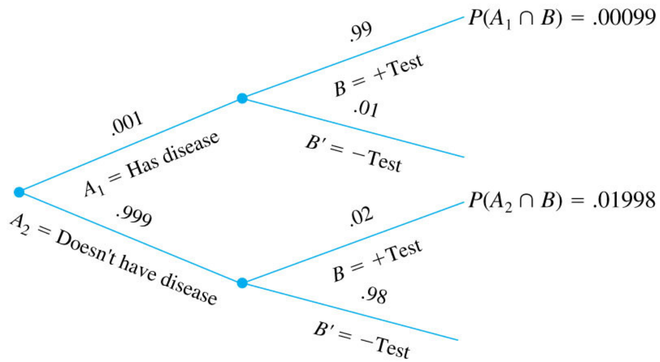
Example 2.30

- **Incidence of a rare disease:** Let

$$A_1 = \{\text{individual has the disease}\}$$
$$A_1 = \{\text{individual does not have the disease}\}$$
$$B_1 = \{\text{positive test result}\}$$

We have

$$P(A_1) = .001, \quad P(A_2) = .001, \quad P(B|A_1) = .99, \quad P(B|A_2) = .02.$$



$$P(B) = .00099 + .01998 = .02097$$
$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{.00099}{.02097} = .047.$$

- The disease is rare and the test only moderately reliable, most positive test results arise from errors rather than from diseased individuals. To get a further increase in the posterior probability, a diagnostic test with much smaller error rates is needed.