Final Exam
Introduction to PDE
MATH 3363-25820 (Fall 2009)

Solutions to Final Exam

This exam has 4 questions, for a total of 40 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Upon finishing PLEASE write and sign your pledge below:

On my honor I have neither given nor received any aid on this exam.

1 Rules

Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks. You are free to use any of the information given in Section 2, without proof, on any question in the exam.

2 Given

You may use the following without proof:

The eigen-solution to

$$X'' + \lambda X = 0, \quad X(0) = 0 = X(L)$$

is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

Orthogonality condition for sines: for any $L > 0$,

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} L/2, & m = n \\ 0 & m \neq n \end{cases}$$

A useful result derived from the Divergence Theorem,

$$\iint_D v\Delta v dV = -\iint_D |\nabla v|^2 dV + \int_{\partial D} v\nabla v \cdot n dS$$

for and 2D or 3D region D with closed boundary ∂D.

3 Questions

10 points 1. A bar with initial temperature profile $f(x) > 0$, with ends held at 0^oC, will cool as $t \to \infty$; and approach a steady-state temperature 0^oC. However, whether or not all parts of the bar start cooling initially depends on the shape of the initial temperature profile. The following heat problem may enable you to discover the relationship:

(a) Solve the heat problem on the interval $0 \le x \le 1$,

$$u_t = u_{xx}, \quad u(0,t) = 0 = u(1,t), \quad u(x,0) = f(x),$$

where

$$f(x) = -\frac{1}{2}\sin 3\pi x + \frac{3}{2}\sin \pi x.$$

Solutoin: Using separation of variables, we let

$$u(x,t) = X(x)T(t)$$

and substitute this into the PDE to obtain

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda$$

where λ is a constant because the left hand side depends only on x and the middle only depends on t.
The Sturm-Liouville problem for $X(x)$ is

$$X'' + \lambda X = 0; \quad X(0) = 0 = X(1)$$

whose solution is (given),

$$X_n(x) = \sin(n\pi x), \quad \lambda_n = n^2\pi^2, \quad n = 1, 2, \ldots$$

The equation

$$T'' = -\lambda T$$

gives the solution

$$T_n(t) = B_n e^{-n^2\pi^2 t}, \quad n = 1, 2, \ldots,$$

Therefore, the solution $u_n(x,t)$, for $n = 1, 2, \ldots$, to the PDE is

$$u_n(x,t) = X_n(x)T_n(t) = B_n \sin(n\pi x)e^{-n^2\pi^2 t}$$

for constants B_n. Summing all $u_n(x,t)$ together gives

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)e^{-n^2\pi^2 t}$$

Imposing the IC gives

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

Multiplying by $\sin(m\pi x)$, for $m = 1, 2, \ldots$, and integrating from $x = 0$ to $x = 1$ gives

$$\int_0^1 f(x)\sin(m\pi x)dx = \sum_{n=1}^{\infty} B_n \int_0^1 \sin(n\pi x)\sin(m\pi x)dx$$

Using the given orthogonality condition gives

$$B_m = 2\int_0^1 f(x)\sin(m\pi x)dx, \quad m = 1, 2, \ldots$$

Since the form of $f(x)$, as given, is already a sine series, from the orthogonality of $\sin(n\pi x)$, we have

$$f(x) = \frac{3}{2}\sin(\pi x) - \frac{1}{2}\sin(3\pi x) \quad \Rightarrow \quad B_1 = \frac{3}{2}, \quad B_3 = -\frac{1}{2}, \quad \text{and } B_n = 0 \text{ for all other n}$$

Therefore,

$$u(x,t) = \frac{3}{2}\sin(\pi x)e^{-\pi^2 t} - \frac{1}{2}\sin(3\pi x)e^{-9\pi^2 t}$$

(b) Show that for some x, $0 < x < 1$, $u_t(x,0)$ is positive (i.e., warming) and for others it is negative (i.e, cooling).
Hint: in Figure 1, $u(x,t_0)$ is plotted for $t_0 = 0, 0.2, 0.5, 1$. You only need to find some x for which u_t is positive/negative.

Solutoin: Differentiating $u(x,t)$ in time gives

$$u_t(x,t) = -\pi^2\left(\frac{3}{2}\sin(\pi x)e^{-\pi^2 t} - \frac{9}{2}\sin(3\pi x)e^{-9\pi^2 t}\right)$$

Setting $t = 0$ gives

$$u_t(x,0) = -\frac{3}{2}\pi^2\left(\sin(\pi x) - 3\sin(3\pi x)\right)$$

Note that

$$u_t\left(\frac{1}{6},0\right) = \frac{15}{4}\pi^2 > 0, \quad u_t\left(\frac{1}{2},0\right) = -6\pi^2 < 0$$

Thus at $x = 1/6$, u_t is positive and for $x = 1/2$, u_t is negative.

(c) How is the sign of $u_t(x,0)$ (i.e., warming/cooling) related to the shape of the initial temperature profile? How is the sign of $u_t(x,t)$, $t > 0$ (i.e., warming/cooling), related to subsequent temperature profiles?

Hint: the PDE gives $u_t = u_{xx}$ and the sign of u_{xx} gives the concavity of $u(x,t)$.

Solutoin: From the PDE,

$$u_t = u_{xx}$$

and hence the sign of $u_t(x,t_0)$, for a fixed $t_0 \geq 0$, gives the concavity of the temperature profile $u(x,t_0)$. Therefore, for some x, $u_t(x,t_0)$ is positive (i.e., warming), we have $u_{xx}(x,t_0) > 0$, thus the profile $u(x,t_0)$ is concave up; for other x, $u_t(x,t_0)$ is negative (i.e., cooling), we have $u_{xx}(x,t_0) < 0$, and the profile $u(x,t_0)$ is concave down. At $t_0 = 0$, the sign of $u_t(x,0)$ (i.e., warming/cooling) give the concavity of the initial temperature profile $u(x,0) = f(x)$.

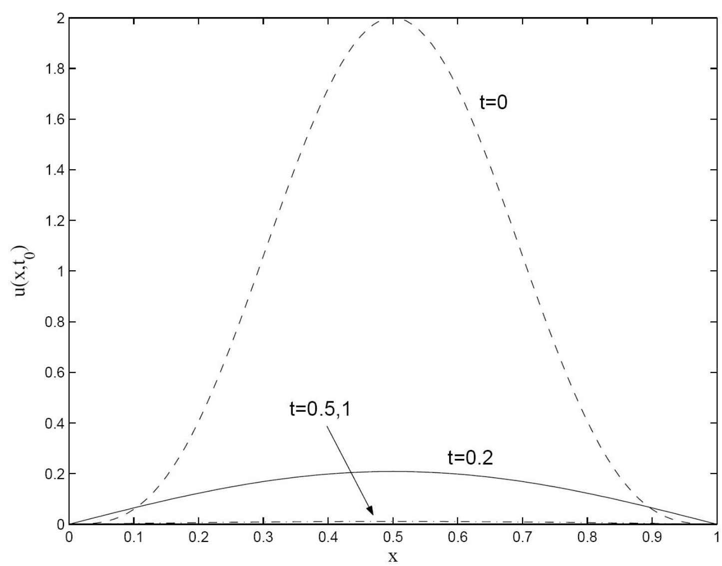

Figure 1. Plots of $u(x,t_0)$ for $t_0 = 0, 0.2, 0.5, 1$

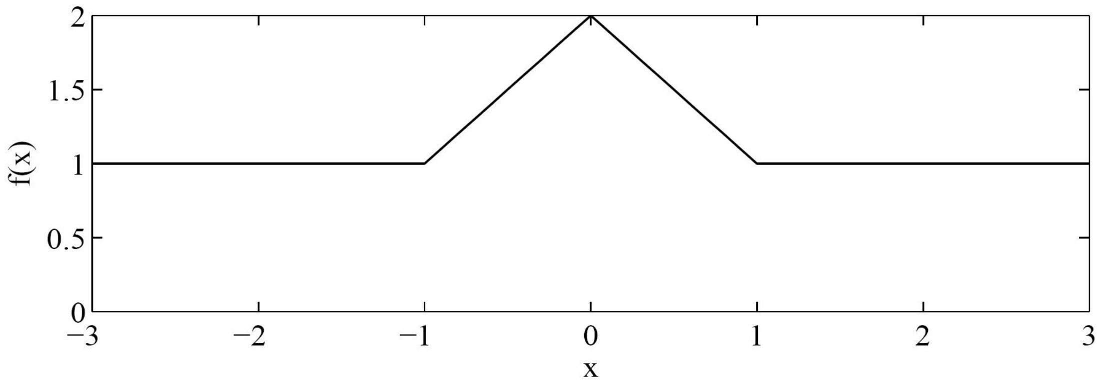

Figure 2. Plot of $f(x)$

10 points

2. Consider the quasi-linear PDE

$$\frac{\partial u}{\partial t} + (1-u)\frac{\partial u}{\partial x} = 0, \quad u(x,0) = f(x)$$

where the initial condition $f(x)$ (shown in Fig. 2) is

$$f(x) = \begin{cases} 1, & |x| > 1 \\ 2 - |x|, & |x| \le 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2 + x, & -1 \le x \le 0 \\ 2 - x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

(a) Find the parametric solution using r as your parameter along a characteristic and s to label the characteristic (i.e. the initial value of x).

Solutoin: To find the parametric solution, we can write the PDE as

$$(1, 1-u, 0) \cdot \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1 \right) = 0.$$

Thus the parametric solution is defined by the ODEs

$$\frac{dt}{dr} = 1, \quad \frac{dx}{dr} = 1 - u, \quad \frac{du}{dr} = 0$$

with initial conditions at $r = 0$,

$$t = 0, \quad x = s, \quad u = u(x,0) = u(s,0) = f(s).$$

Integrating the ODEs and imposing the ICs gives

$$
\begin{aligned}
t(r;s) &= r \\
u(r;s) &= f(s) \\
x(r;s) &= (1-f(s))r + s = (1-f(s))t + s
\end{aligned}
$$

(b) At what time t_{sh} and location x_{sh} does your parametric solution break down?

Solutoin: To find the time t_{sh} and position x_{sh} when and where a shock first forms, we find the Jacobian:

$$
J = \frac{\partial(x,t)}{\partial(r,s)} = \det\begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = x_r t_s - x_s t_r = -x_s = f'(s)t - 1
$$

Shocks occur (the solution breaks down) where $J = 0$, i.e. where

$$
t = \frac{1}{f'(s)}
$$

The first shock occurs at

$$
t_{\text{sh}} = \min\left(\frac{1}{f'(s)}\right) = \frac{1}{\max(f'(s))}
$$

Since $\max(f'(s)) = 1$, we have

$$
t_{\text{sh}} = \frac{1}{\max(f'(s))} = 1.
$$

Any of the characteristics where $f'(s) = \max(f'(s)) = 1$ can be used to find the location of the shock at $t_{\text{sh}} = 1$. For e.g., with $s = -1/2$, the location of the shock at $t_{\text{sh}} = 1$ is

$$
x_{\text{sh}} = \left(1 - f\left(-\frac{1}{2}\right)\right)1 - \frac{1}{2} = \left(1 - \left(2 - \frac{1}{2}\right)\right)1 - \frac{1}{2} = -1.
$$

Any other value of s where $f'(s) = \max(f'(s)) = 1$ will give the same x_{sh}.

(c) For each of $s = -1, 0, 1$, write down the characteristic x as a functions of t and plot the characteristic in the space-time plane (xt).

Solutoin: The $s = -1, 0, 1$ characteristics are given by

$$s = -1: \quad x = (1 - f(-1))t - 1 = -1$$

$$s = 0: \quad x = (1 - f(0))t + 0 = -t$$

$$s = 1: \quad x = (1 - f(1))t + 1 = 1$$

These are plotted in the figure below.

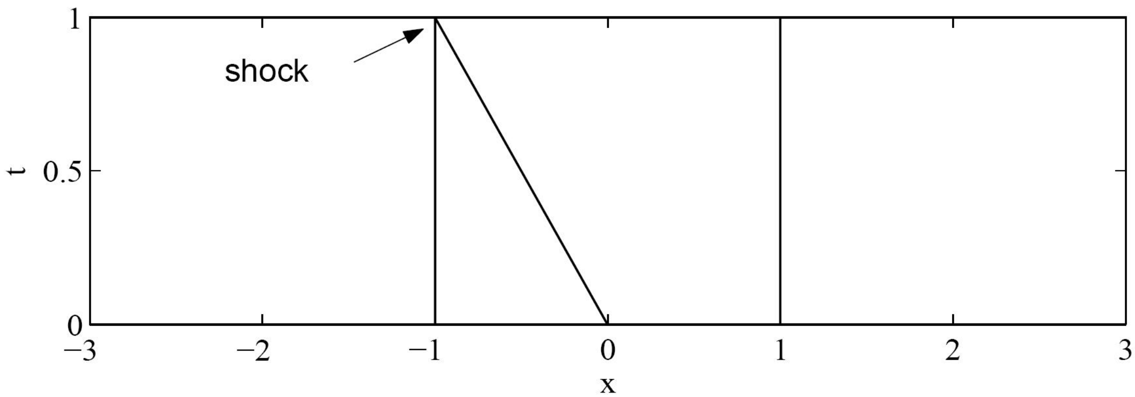

Figure 2.c Plot of characteristics

(d) The tables below are useful as a plotting aid. Fill in the tables using your result from (a) to obtain u and x at the s-values listed at time $t = 1/2, 1$.

$$t = \tfrac{1}{2} \qquad \begin{array}{rccc} s = & -1 & 0 & 1 \\ u = & & & \\ x = & & & \end{array}$$

$$t = 1 \qquad \begin{array}{rccc} s = & -1 & 0 & 1 \\ u = & & & \\ x = & & & \end{array}$$

Solutoin:

$$t = \tfrac{1}{2} \qquad \begin{array}{rrrr} s = & -1 & 0 & 1 \\ u = & 1 & 2 & 1 \\ x = & -1 & -1/2 & 1 \end{array}$$

$$t = 1 \qquad \begin{array}{rrrr} s = & -1 & 0 & 1 \\ u = & 1 & 2 & 1 \\ x = & -1 & -1 & 1 \end{array}$$

(e) Illustrate the time evolution of the solution by sketching the ux-profiles $u(x,t)$ for $t = 0,\ 1/2,\ 1$.

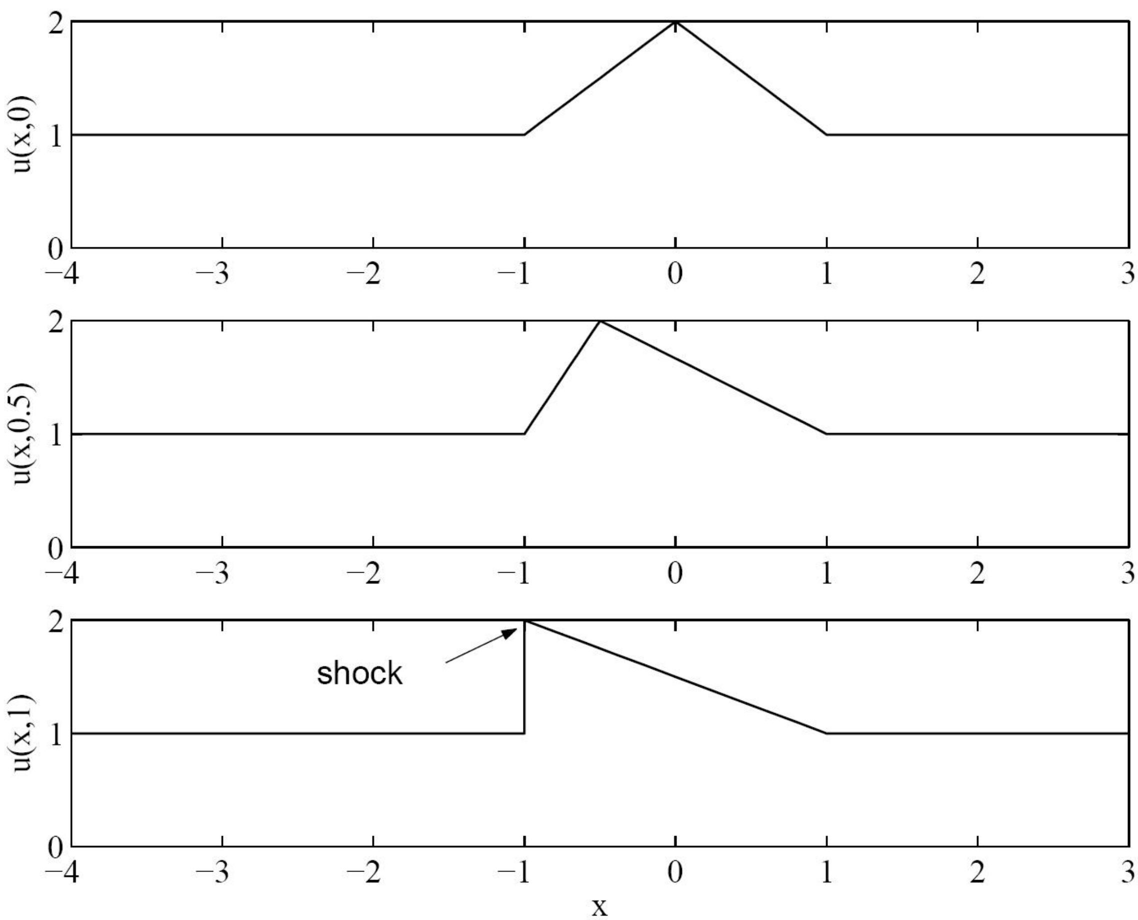

Figure 2.d Plot of $u(x, t_0)$ for $t_0 = 0$, $1/2$ and 1.

Solutoin: A plot of $u(x, 1/2)$ is made by plotting the three points (x, u) from the table for $t = 1/2$ and connecting the dots (see middle plot in the figure above). Similarly, $u(x, t_{\text{sh}}) = u(x, 1)$ is plotted in the last plot.

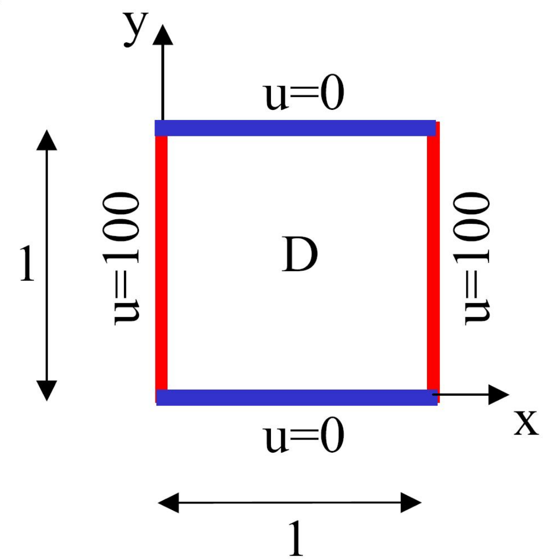

Figure 3. Unit square D and Inhomogeneous BCs

10 points 3. (a) Solve the Heat Problem on the 2D unit square $D = \{(x, y) : 0 \leq x, y \leq 1\}$

$$u_t = \Delta u, \quad (x, y) \in D, \quad t > 0,$$

subject to inhomogeneous BCs

$$u(x, y, t) = \begin{cases} 100, & x = 0 \text{ or } 1, \quad 0 < y < 1, \\ 0, & \text{otherwise on } \partial D \end{cases}$$

and initial condition

$$u(x, y, 0) = 0, \quad (x, y) \in D.$$

Hint: first derive the steady-state (equilibrium) solution u_E, set $v = u - u_E$, then transform the given heat problem for u into the homogeneous heat problem for v and solve it for v.

Solutoin: First, let $u_E(x, y)$ be the steady-state (equilibrium) solution. Then u_E satisfies Laplace's equation on the 2D unit square

$$\Delta u_E = 0, \quad (x, y) \in D,$$

subject to 2 nonhomogeneous BCs and 2 homogeneous BCs

$$u_E(x, y) = \begin{cases} 100, & x = 0 \text{ or } 1, \quad 0 < y < 1, \\ 0, & \text{otherwise on } \partial D \end{cases}$$

Apply the principle of superposition and break the problem into two problems each having one nonhomogeneous BC. Let

$$u_E(x,y) = u_E^1(x,y) + u_E^2(x,y)$$

where each $u_E^i(x,y)$ satisfies Laplace's equation with one nonhomogeneous BC and the related three homogeneous BCs,

$$\begin{cases} \Delta u_E^1 = 0, \quad (x,y) \in D, \\ u_E^1(x,y) = \begin{cases} 100, \quad x = 0, \quad 0 < y < 1, \\ 0, \quad \text{otherwise on } \partial D \end{cases} \end{cases} \qquad \begin{cases} \Delta u_E^2 = 0, \quad (x,y) \in D, \\ u_E^2(x,y) = \begin{cases} 100, \quad x = 1, \quad 0 < y < 1, \\ 0, \quad \text{otherwise on } \partial D \end{cases} \end{cases}$$

The method to solve for any of the $u_E^i(x,y)$ is the same; only certain details differ. Furthermore, by the argument of symmetry, we have $u_E^2(x,y) = u_E^1(1-x,y)$, thus we only need to solve for $u_E^1(x,y)$.
We proceed via separation of variables: $u_E^1(x,y) = X(x)Y(y)$, so that the PDE becomes

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda$$

where λ is constant since the l.h.s. depends only on x and the middle only on y. The 3 homogenerous BCs are

$$Y(0) = Y(1) = 0, \quad X(1) = 0,$$

and

$$X(0)Y(y) = 100, \quad 0 < y < 1,$$

We first solve for $Y(y)$, since we have 2 easy BCs:

$$Y'' + \lambda Y = 0, \quad 0 < y < 1, \qquad Y(0) = Y(1) = 0$$

The non-trivial solutions (given) are

$$Y_n(y) = \sin(n\pi y), \quad \lambda_n = n^2\pi^2, \quad n = 1, 2, \cdots.$$

Now we consider $X(x)$:

$$X'' - n^2\pi^2 X = 0$$

and hence

$$X(x) = c_1 e^{n\pi x} + c_2 e^{-n\pi x}$$

An equivalent and more convenient way to write this is

$$X(x) = c_3 \sinh n\pi(1-x) + c_4 \cosh n\pi(1-x)$$

Imposing the BC at $x = 1$ gives
$$X(1) = c_4 = 0$$
and hence
$$X(x) = c_3 \sinh n\pi(1-x)$$
Thus the solution is
$$u_E^1(x,y) = \sum_{n=1}^{\infty} B_n \sinh\big(n\pi(1-x)\big)\sin(n\pi y)$$
which satisfies the PDE and the 3 homogenerous BCs on $x = 1$ and $y = 0, 1$. Also, from the nonhomogenerous BC on $x = 0$, we have
$$u_E^1(0,y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi)\sin(n\pi y) = 100, \quad 0 < y < 1$$
Multiplying both sides by $\sin(m\pi y)$ and integrating in y gives
$$\sum_{n=1}^{\infty} B_n \sinh(n\pi)\int_0^1 \sin(n\pi y)\sin(m\pi y)dy = 100\int_0^1 \sin(m\pi y)dy$$
From the orthogonality of sin's (given), we have
$$B_m \sinh(m\pi)\frac{1}{2} = 100\int_0^1 \sin(m\pi y)dy = 100\frac{1-\cos(m\pi)}{m\pi} = 100\frac{1-(-1)^m}{m\pi}$$
Thus
$$B_m = \frac{200}{m\pi\sinh(m\pi)}\big(1-(-1)^m\big) = \begin{cases} 0, & m \text{ even} \\ \frac{400}{m\pi\sinh(m\pi)}, & m \text{ odd} \end{cases}$$
In other words,
$$B_{2n} = 0, \quad B_{2n-1} = \frac{400}{(2n-1)\pi\sinh\big((2n-1)\pi\big)}, \quad n = 1, 2, \cdots .$$
and the solution becomes
$$u_E^1(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{\sinh\big((2n-1)\pi(1-x)\big)}{(2n-1)\sinh\big((2n-1)\pi\big)}\sin((2n-1)\pi y)$$
Thus the steady-state (equilibrium) solution is
$$\begin{aligned} u_E(x,y) &= u_E^1(x,y) + u_E^2(x,y) = u_E^1(x,y) + u_E^1(1-x,y) \\ &= \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{\sinh\big((2n-1)\pi(1-x)\big) + \sinh\big((2n-1)\pi x\big)}{(2n-1)\sinh\big((2n-1)\pi\big)}\sin((2n-1)\pi y) \end{aligned}$$

To solve the transient problem, we define

$$v(x, y, t) = u(x, y, t) - u_E(x, y)$$

so that v satisfies

$$\begin{aligned}
&v_t = \Delta v, \quad (x, y) \in D, \quad t > 0, \\
&v = 0, \qquad (x, y) \in \partial D, \quad t > 0, \\
&v(x, y, 0) = -u_E(x, y), \quad (x, y) \in D.
\end{aligned}$$

This is the heat problem with homogeneous PDE and BCs. We separate variables as

$$v(x, y, t) = \phi(x, y)T(t)$$

The 2D heat equation implies

$$\frac{T'}{T} = \frac{\Delta \phi}{\phi} = -\lambda$$

where λ is a constant since the l.h.s. depends solely on t and the middle depends solely on x. On the boundaries,

$$\phi(x, y) = 0, \quad (x, y) \in \partial D$$

The Sturm-Liouville Problem for ϕ is

$$\Delta \phi + \lambda \phi = 0, \quad (x, y) \in D; \qquad \phi(x, y) = 0, \quad (x, y) \in \partial D.$$

or more precisely, on the unit square D

$$\begin{aligned}
&\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \lambda \phi = 0, \quad 0 < x, y < 1, \\
&\phi(0, y) = \phi(1, y) = 0 \quad 0 < y < 1; \qquad \phi(x, 0) = \phi(x, 1) = 0 \quad 0 < x < 1
\end{aligned}$$

We employ separation of variables again, this time in x and y: substituting $\phi(x, y) = X(x)Y(y)$ into the PDE and dividing by $X(x)Y(y)$ gives

$$\frac{Y''}{Y} + \lambda = -\frac{X''}{X} = \mu$$

where μ is constant since the l.h.s. depends only on y and the middle only depends on x. The homogeneous BCs imply

$$X(0) = X(1) = Y(0) = Y(1) = 0$$

The problem for $X(x)$ is the 1D Sturm-Liouville problem

$$X'' + \mu X = 0, \quad 0 < x < 1; \quad X(0) = X(1) = 0$$

The non-trivial solutions (given) are

$$X_n(x) = \sin(n\pi x), \quad \mu_n = n^2\pi^2, \quad n = 1, 2, \cdots .$$

The problem for $Y(y)$ is

$$Y'' + \nu Y = 0, \quad 0 < y < 1, \qquad Y(0) = Y(1) = 0$$

where $\nu = \lambda - \mu$. The non-trivial solutions (given) are

$$Y_n(x) = \sin(n\pi y), \quad \nu_n = n^2\pi^2, \quad n = 1, 2, \cdots .$$

The eigen-solution of the 2D Sturm Liouville problem is

$$\phi_{mn}(x, y) = X_m(x)Y_n(y) = \sin(m\pi x)\sin(n\pi y), \quad m, n = 1, 2, \cdots ,$$

with eigenvalue

$$\lambda_{mn} = \mu_m + \nu_n = \pi^2(m^2 + n^2)$$

The problem for $T(t)$ is

$$T' = -\lambda T$$

with solution

$$T_{mn}(t) = B_{mn}e^{-\lambda_{mn}t}$$

Thus

$$v_{mn}(x, y, t) = \phi_{mn}(x, y)T_{mn}(t) = B_{mn}\sin(m\pi x)\sin(n\pi y)e^{-\lambda_{mn}t}$$

To satisfy the initial condition, we sum over all m, n to obtain the solution, in general form,

$$v(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{mn}(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin(m\pi x)\sin(n\pi y)e^{-\lambda_{mn}t}$$

Setting $t = 0$ and imposing the initial condition $v(x, y, 0) = -u_E(x, y)$ gives

$$v(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin(m\pi x)\sin(n\pi y) = -u_E(x, y)$$

Multiplying both sides by $\sin(m'\pi x)\sin(n'\pi y)$ and integrating in x and y gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\int_0^1 \sin(m\pi x)\sin(m'\pi x)dx \int_0^1 \sin(n\pi y)\sin(n'\pi y)dy$$

$$= -\int_0^1\int_0^1 u_E(x, y)\sin(m\pi x)\sin(n'\pi y)dxdy$$

From the orthogonality of sin's (given), we have

$$B_{mn} = -4 \int_0^1 \int_0^1 u_E(x, y) \sin(m\pi x) \sin(n\pi y) dx dy$$

(b) Prove the solution to (a) is unique.

Hint: you'll need to use a result derived from the Divergence Theorem (on the given page).

Solutoin: Take two solutions u_1 and u_2. Define the difference $w = u_1 - u_2$. Note that w satisfies

$$w_t = \Delta w, \quad (x, y) \in D, \quad t > 0,$$

subject to the homogeneous BCs

$$u(x, y, t) = 0, \qquad (x, y) \in \partial D, \quad t > 0,$$

and the homogeneous initial condition

$$w(r, \theta, 0) = 0, \quad (x, y) \in D.$$

Define the mean square difference between solutions,

$$E(t) = \iint_D w(t)^2 dV \geq 0$$

Differentiate in time and apply the Divergence Theorem

$$\begin{aligned} \frac{dE(t)}{dt} &= \iint_D 2ww_t dV = 2 \iint_D w \Delta w dV \\ &= -2 \iint_D |\nabla w|^2 dV + 2 \int_{\partial D} w \nabla w \cdot n dS \end{aligned}$$

But $w = 0$ on the boundary ∂D, so that

$$\frac{dE(t)}{dt} = -2 \iint_D |\nabla w|^2 dV \leq 0$$

Note that at $t = 0$

$$E(0) = \iint_D w(0)^2 dV = 0$$

Thus, $E(t)$ is a non-negative, non-increasing function , that starts at zero. Hence $E(t) = 0$ for all time, which implies by continuity that $w(r, \theta, t) = 0$ for all r, θ, t. Hence $v_1 = v_2$, and the solution to 2 is unique.

10 points 4. Consider the Boundary Value Problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D \\ u &= f \quad \text{on } \partial D \end{aligned}$$

where D is a simply-connected 2d region with piecewise smooth boundary ∂D.

(a.1) State the Maximum Principle for u on D.

Solutoin: Maximum principle: solution to laplace's equation takes min/max on boundary

$$\min_{\partial D} f \le u \le \max_{\partial D} f$$

(a.2) If $f = 25$ at each point on the boundary ∂D, what is u in D? Explain your answer.

Solutoin: If $f = 25$,

$$\min_{\partial D} f = \max_{\partial D} f = 25 \quad \Rightarrow \quad 25 \le u \le 25$$

Thus $u = 25$.

(b.1) Now let D be the disc of radius R centered at the origin,

$$D = \{(x, y) : x^2 + y^2 \le R^2\}.$$

Name and state (without proof) another property of u which gives the value of u at the center of the disc in terms of the values of u on the boundary

$$\partial D = \{(x, y) : x^2 + y^2 = R^2\}.$$

Solutoin: Mean value property: Let $u(x, y)$ be solution to laplace's equation. Then, the value of u at any point $(x_0, y_0) \in D$ equals the mean value of u around any circle of radius r centered at (x_0, y_0) and contained in D,

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\, d\theta$$

Applying the mean value property gives the value of u at the center of the disc in terms of the values of u on the boundary of the disc

$$u(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta$$

(b.2) Use this result to find $u(0,0)$ if on the boundary, u takes the values

$$u(R,\theta) = \begin{cases} 25, & -\pi/2 \leq \theta \leq \pi/2, \\ 26, & \pi/2 \leq \theta \leq \pi, \\ 24, & \pi \leq \theta \leq 3\pi/2. \end{cases}$$

Solutoin:

$$\begin{aligned} u(0,0) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \\ &= \frac{1}{2\pi}\big(25\pi + 26\pi/2 + 24\pi/2\big) \\ &= \frac{1}{2}(25 + 13 + 12) \\ &= 25 \end{aligned}$$