Section 1.4 Problem Notes

1.4.12.

(I) You can't assume a certain form for W. I.e., you can't prove this by first assuming $W = \{w_1, ..., w_p\}$, because in general when W is a subspace, it will have infinitely many vectors.

(II) A good proof has a clear line of thought and logically proceeds from one step to the next. Here's an example of the process behind this proof.

(i) Write down what statement you are trying to prove: "$W \subset V$ is a subspace of V iff. $\text{Span}(W) = W$."

(ii) This is an iff. statement, so we have two directions to prove: "First assume W is a subspace of V."

(iii) We want to show that $\text{Span}(W) = W$. Which requires showing $\text{Span}(W) \subseteq W$ and $\text{Span}(W) \supseteq W$. We know the span of a set is the set of linear combinations, hence we get the second containment by defintion "$\text{Span}(W) \supseteq W$ by the definition of the span of the set; since the span of W contains all linear combinations, it will in particular contain every w in W."

(iv) To finish the first direction of the iff. we need to show the other containment, this is where we need the assumption that W is a subspace. "Let $v \in \text{Span}(W)$. Then $\exists w_1, ..., w_k \in W$ and $a_1, ..., a_k \in \mathbb{F}$ such that $v = a_1w_1 + ... + a_kw_k$. But this is a linear combination of elements in W and since W is a subspace it contains all linear combinations of its elements, hence $v \in W$ and so $\text{Span}(W) \subseteq W$. "

(v) Wrap up the first defintion with a sum of what you proved: "Therefore $\text{Span}(W) = W$ when W is a subspace."

(vi) Now we do the other direction: "Next, assume $\text{Span}(W) = W$."

(vii) This direction is by a fact about spans and the equality. "Since the span of a set is a subspace of V and $\text{Span}(W) = W$, then W must be a subspace, proving the other direction."

(viii) So, altogether we get a solid proof of this statement:

Proposition 0.1. *$W \subset V$ is a subspace of V iff. Span$(W) = W$.*

Proof. ($\implies$) First assume W is a subspace of V. $\text{Span}(W) \supseteq W$ by the definition of the span of the set; since the span of W contains all linear combinations, it will in particular contain every w in W.
Let $v \in \text{Span}(W)$. Then $\exists w_1, ..., w_k \in W$ and $a_1, ..., a_k \in \mathbb{F}$ such that $v = a_1w_1 + ... + a_kw_k$. But this is a linear combination of elements in W and since W is a subspace it contains all linear combinations of its elements, hence $v \in W$ and so $\text{Span}(W) \subseteq W$.
Therefore $\text{Span}(W) = W$ when W is a subspace.
($\impliedby$) Next, assume $\text{Span}(W) = W$. Since the span of a set is a subspace of V and $\text{Span}(W) = W$, then W must be a subspace, proving the other direction. □

1.4.15.

(I) Some of you attempted to prove by contradiction, which is fine, but the negative of $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_1) \cap \text{Span}(S_2)$ is NOT $\text{Span}(S_1 \cap S_2) \supset \text{Span}(S_1) \cap \text{Span}(S_2)$. "$\subseteq$" does not work like an inequality. The negative of $A \subseteq B$ would be the assumption that there is something inside of A and is not inside of B.

1.5.9.

(I) When you assume the two vectors u and v are linearly independent, then there exists constants $a, b \in \mathbb{F}$ which aren't both zero such that $au + bv = 0$. This means either[1] $a \neq 0$ and we can express u as $\frac{-b}{a}v$ or $b \neq 0$ and we can express v as $v = \frac{-a}{b}u$.

(II) Don't make extra assumptions that the problem doesn't give. For example, the two vectors here do not need to be distinct. So, you don't get to assume that they are distinct unless you prove why. (Most people who made this assumption gave a proof that works just as well for the case where they are the same, making the assumption superfluous.)

1.5.13.

(I) When you are showing linear independence you can't put restrictions on the coefficients without reasoning why. For example, if you first assume $(a+b)v_1 + (a-b)v_2 = 0$ and then show $a = b = 0$ then you still haven't shown v_1 and v_2 to be linearly independent. All you've don't is show that when $\lambda_1 v_1 + \lambda v_2 = 0$ implies $\lambda_1 = \lambda_2 = 0$ when $\lambda_1 = a + b$ and $\lambda_2 = a - b$. However, you don't know that λ_1 and λ_2 can always be written like this. It's a subtle but important obstacle.[2]

(II) A good way to tell whether or not you did a problem correctly is to try and pinpoint where you used the different assumptions. In this case, make sure you used the fact that the characteristic of the field is not 2.

(III) Be wary of writing something like $\frac{a+b}{2}$ when you are working in a general field. For $\mathbb{F} = \mathbb{R}$, this just means to divide the sum of a and b by 2. But for a more general field, dividing by 2 doesn't necesarily make sense. You then have to interpret $\frac{a+b}{2}$ as "the element q such that $2q = a + b$." There's an issue remaining though. Why does such a q exist?

(IV) With the above in mind, a safe way to do $\{u - v, u + v\}$ linearly independent implies $\{u, v\}$ linearly independent is:

Assume $au + bv = 0$ for some $a, b \in \mathbb{F}$. Then $2au + 2bv = 2 \cdot 0 = 0$. Since $bu - bu = 0$ and $av - av = 0$ we can add the three equations together to get $2au + bu - bu + av - av + 2bv = 0$. Rearranging gives

$$0 = [au + bu + av + bv] + [au - bu - av + bv] = (a+b)(u+v) + (a-b)(u-v).$$

Since $\{u - v, u + v\}$ is linearly independent, $a + b = a - b = 0$ which implies $2a = 0$ and $2b = 0$ and since $\mathbb{F}$ is not characteristic 2, $a = b = 0$.

[1]You could also say "Without loss of generality, $a \neq 0$" and just do the first case if you are familiar with that notion.

[2]One way to surmount this might be to show that any two elements of a field can be written in that way, but this is more difficult than proving this along a different method. Also note that when your field has characteristic 2 it's false: there's a field with four elements where every element is its own additive inverse: $x = -x$, for all $x \in \mathbb{F}$. Thus $a + b = a - b$ for any $a, b \in \mathbb{F}$. Thus when we have two distinct elements in this field we can't possible have one look like $a - b$ and the other $a + b$.