Section 2.3 Problem Notes

2.3.12.

(1) Since these facts are used frequently in other proofs, I shall give a detalied proof below:
Let $T : V \to W$ and $U : W \to Z$ be linear maps, so then $UT : V \to Z$ is also a linear map.

(a) Assume UT is one-to-one. Since UT is linear, this means it has trivial null space, and in other words if $UT(v) = 0$ then $v = 0$.
Since T is a linear map, we show that it is one-to-one by showing that its nullspace is trivial: Assume $T(v) = 0$ for some $v \in V$. Then $UT(v) = U(T(v)) = U(0) = 0$[1] since U is linear. Hence $v = 0$ since UT is one-to-one. Therefore T is one-to-one.
For one example of UT one-to-one but not U, let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the inclusion map $T(x, y) = (x, y, 0)$ and $U : \mathbb{R}^3 \to \mathbb{R}$ be projection onto the first coordinate: $U(x, y, z) = (x, y)$. Then $UT(x, y) = U(x, y, 0) = (x, y)$ so $UT : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map and thus definitely injective, even though U isn't[2] So, what's happening here is U is one-to-one on the range of T, but sends many elements to the same place outside the range of T.

(b) Assume UT is onto. To show that U is onto, let $c \in Z$ and we want to find a $b \in W$ such that $U(b) = c$. Since UT is onto, there exists $a \in V$ such that $UT(a) = c$. This means $U(T(a)) = c$ hence letting $b = T(a)$ gives us a U-preimage for arbitrary $c \in Z$ and thus U is onto.
For one example of UT onto but not T, look no further than the example above. T isn't onto because it doesn't map to anything with nonzero third coordinate, but UT is onto.

(c) If both U and T are one-to-one and onto, then since they are linear they are isomorphisms.[3]
If $UT(v) = 0$ then $U(T(v)) = 0$, hence $T(v)$ is in the kernel of U. But, U is one-to-one and linear, so $T(v) = 0$. Again we have T is one-to-one and linear, so $v = 0$, hence UT is linear with trivial kernel, thus is one-to-one.
Let $c \in Z$. Then since U is onto there is a $b \in W$ such that $U(b) = c$. Since T is onto there is an $a \in V$ such that $T(a) = U(b)$. Hence $UT(a) = U(T(a)) = U(b) = c$ and so UT is onto.

(2) A few people tried a proof by contrapositive here, but didn't have the right negation.

Section 2.4 Problem Notes

2.4.15. Appealing to 2.1.14(c) doesn't work on two accounts: (i) it doesn't show both directions (ii) we haven't done that exercise.

Also note that this problem was covered in the exam review session I held last Monday.

Section 2.5 Problem Notes

2.5.2. The idea for these is to write the matrix representation of the identity map from β' coordinates to β coordinates: $[I]_{\beta'}^{\beta}$.

So, you take each basis vector in β' and express it as a linear combination of the β basis elements. These coefficients will give you a column of your change of basis matrix.

[1] Writing out this equation is the clearest way to demonstrate this, because it tells us why $UT(v) = 0$ — something many proofs left out

[2] $\text{Nul}(U) = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\} = \{0\} \times \{0\} \times \mathbb{R}$.

[3] We are showing that a composition of isomorphisms is an isomorphism.

(a) This one is the easiest: What is (a_1, a_2) in the standard basis: $(a_1, a_2) = a_1e_1 + a_2e_2$. Thus the first column is $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, we similarly find the second column to get $[I]_{\beta'}^{\beta} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

(b) One way to do this is to solve $(0, 10) = a_1(-1, 3) + a_2(2, -1)$.

Since β' is just scaling and reordering the standard basis, it's easy to find the change of coordinates in the opposite direction first: $[I]_{\beta}^{\beta'} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$

The direction we want is

$$[I]_{\beta'}^{\beta} = ([I]_{\beta}^{\beta'})^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}^{-1}$$

I find this to be an easy way to do these problems, because finding the inverse of a 2×2 matrix has an easy trick:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus

$$[I]_{\beta'}^{\beta} = \frac{1}{\frac{1}{50}(6-1)} \begin{bmatrix} \frac{2}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{3}{10} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

(c) Again, it's easier to find $[I]_{\beta}^{\beta'} = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$ and invert it to get

$$[I]_{\beta'}^{\beta} = \frac{1}{-1} \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}$$

(d) Finally, use the following identity (sort of a cancellation rule for basis change that can make calculations easy when the inverse is easy to calculate):

$$[I]_{\beta'}^{\beta} = [I]_{std}^{\beta}[I]_{\beta'}^{std}$$

where *std* refers to the standard basis.

Then we find $[I]_{\gamma}^{std}$ pretty easily, and invert it where necessary:

$$[I]_{\beta'}^{\beta} = [I]_{std}^{\beta}[I]_{\beta'}^{std} = ([I]_{\beta}^{std})^{-1}[I]_{\beta'}^{std} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -4 \\ 1 & 1 \end{bmatrix}$$

So,

$$[I]_{\beta'}^{\beta} = \frac{1}{-2} \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & 4 \end{bmatrix}$$

Of course, you can also do this by solving

$$(2, 1) = a_1(-4, 3) + a_2(2, -1)$$
$$(-4, 1) = b_1(-4, 3) + b_2(2, -1)$$

And then $[I]_{\beta'}^{\beta} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, but I personally find the first way a little more enlightening.