Math 4377/6308 Advanced Linear Algebra

Chapter 1 Review and Solution to Problems

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Every vector space has a unique additive identity.

Proof. Suppose there are two additive identities 0 and $0'$. Then

$$0' = 0 + 0' = 0,$$

where the first equality holds since 0 is an identity and the second equality holds since $0'$ is an identity. Hence $0 = 0'$, proving that the additive identity is unique. $\square$

Every $v \in V$ has a unique additive inverse.

Proof. Suppose w and w' are additive inverses of v so that $v + w = 0$ and $v + w' = 0$. Then

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

Hence $w = w'$, as desired. $\square$

$0v = 0$ for all $v \in V$.

Proof. For $v \in V$, we have by distributivity that

$$0v = (0 + 0)v = 0v + 0v.$$

Adding the additive inverse of $0v$ to both sides, we obtain

$$0 = 0v - 0v = (0v + 0v) - 0v = 0v.$$

□

$a0 = 0$ for all $a \in F$.

$$a0 = a(0+0) = a0 + a0.$$

Adding the additive inverse of $a0$ to both sides, we obtain $0 = a0$, as desired. $\square$

$(-1)v = -v$ for every $v \in V$.

Proof. For $v \in V$, we have

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0,$$

which shows that $(-1)v$ is the additive inverse $-v$ of v. $\square$

The list of vectors $(v_1, \cdots, v_m)$ is linearly independent if and only if every $v \in \operatorname{span}(v_1, \cdots, v_m)$ can be uniquely written as a linear combination of $(v_1, \cdots, v_m)$.

("$\Longrightarrow$") Assume that $(v_1, \ldots, v_m)$ is a linearly independent list of vectors. Suppose there are two ways of writing $v \in \operatorname{span}(v_1, \ldots, v_m)$ as a linear combination of the v_i:

$$v = a_1 v_1 + \cdots a_m v_m,$$
$$v = a_1' v_1 + \cdots a_m' v_m.$$

Subtracting the two equations yields $0 = (a_1 - a_1')v_1 + \cdots + (a_m - a_m')v_m$. Since $(v_1, \ldots, v_m)$ is linearly independent, the only solution to this equation is $a_1 - a_1' = 0, \ldots, a_m - a_m' = 0$, or equivalently $a_1 = a_1', \ldots, a_m = a_m'$.

("$\Longleftarrow$") Now assume that, for every $v \in \operatorname{span}(v_1, \ldots, v_m)$, there are unique $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$v = a_1 v_1 + \cdots + a_m v_m.$$

This implies, in particular, that the only way the zero vector $v = 0$ can be written as a linear combination of $v_1, \ldots, v_m$ is with $a_1 = \cdots = a_m = 0$. This shows that $(v_1, \ldots, v_m)$ are linearly independent. $\square$

Linear Dependence Lemma

If $(v_1, \cdots, v_m)$ is linearly dependent and $v_1 \neq 0$, then there exists $j \in \{2, \cdots, m\}$ such that the following two conditions hold.

1. $v_j \in \mathrm{span}(v_1, \cdots, v_{j-1})$.
2. If v_j is removed from $(v_1, \cdots, v_m)$, then $\mathrm{span}(v_1, \cdots, \hat{v}_j, \cdots, v_m) = \mathrm{span}(v_1, \cdots, v_m)$.

Proof. Since $(v_1, \ldots, v_m)$ is linearly dependent there exist $a_1, \ldots, a_m \in \mathbb{F}$ not all zero such that $a_1 v_1 + \cdots + a_m v_m = 0$. Since by assumption $v_1 \neq 0$, not all of $a_2, \ldots, a_m$ can be zero. Let $j \in \{2, \ldots, m\}$ be largest such that $a_j \neq 0$. Then we have

$$v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}, \tag{5.1}$$

which implies Part 1.

Let $v \in \mathrm{span}(v_1, \ldots, v_m)$. This means, by definition, that there exist scalars $b_1, \ldots, b_m \in \mathbb{F}$ such that

$$v = b_1 v_1 + \cdots + b_m v_m.$$

The vector v_j that we determined in Part 1 can be replaced by Equation (5.1) so that v is written as a linear combination of $(v_1, \ldots, \hat{v}_j, \ldots, v_m)$. Hence, $\mathrm{span}(v_1, \ldots, \hat{v}_j, \ldots, v_m) = \mathrm{span}(v_1, \ldots, v_m)$. $\square$

Let $(v_1, \cdots, v_m)$ be a linearly independent list of vectors that spans V, and let $(w_1, \cdots, w_n)$ be any list that spans V. Then $m \leq n$.

Proof. The proof uses the following iterative procedure: start with an arbitrary list of vectors $\mathcal{S}_0 = (w_1, \ldots, w_n)$ such that $V = \operatorname{span}(\mathcal{S}_0)$. At the k^{th} step of the procedure, we construct a new list $\mathcal{S}_k$ by replacing some vector w_{j_k} by the vector v_k such that $\mathcal{S}_k$ still spans V. Repeating this for all v_k then produces a new list $\mathcal{S}_m$ of length n that contains each of $v_1, \ldots, v_m$, which then proves that $m \leq n$. Let us now discuss each step in this procedure in detail.

Step 1. Since $(w_1, \ldots, w_n)$ spans V, adding a new vector to the list makes the new list linearly dependent. Hence $(v_1, w_1, \ldots, w_n)$ is linearly dependent. By Lemma 5.2.7, there exists an index j_1 such that

$$w_{j_1} \in \operatorname{span}(v_1, w_1, \ldots, w_{j_1 - 1}).$$

Hence $\mathcal{S}_1 = (v_1, w_1, \ldots, \hat{w}_{j_1}, \ldots, w_n)$ spans V. In this step, we added the vector v_1 and removed the vector w_{j_1} from $\mathcal{S}_0$.

Step k**.** Suppose that we already added $v_1, \dots, v_{k-1}$ to our spanning list and removed the vectors $w_{j_1}, \dots, w_{j_{k-1}}$ in return. Call this list $\mathcal{S}_{k-1}$, and note that $V = \text{span}(\mathcal{S}_{k-1})$. Add the vector v_k to $\mathcal{S}_{k-1}$. By the same arguments as before, adjoining the extra vector v_k to the spanning list $\mathcal{S}_{k-1}$ yields a list of linearly dependent vectors. Hence, by Lemma 5.2.7, there exists an index j_k such that $\mathcal{S}_{k-1}$ with v_k added and w_{j_k} removed still spans V. The fact that $(v_1, \dots, v_k)$ is linearly independent ensures that the vector removed is indeed among the w_j. Call the new list $\mathcal{S}_k$, and note that $V = \text{span}(\mathcal{S}_k)$.

The final list $\mathcal{S}_m$ is $\mathcal{S}_0$ but with each $v_1, \dots, v_m$ added and each $w_{j_1}, \dots, w_{j_m}$ removed. Moreover, note that $\mathcal{S}_m$ has length n and still spans V. It follows that $m \leq n$. □

Basis Reduction Theorem

If $V = \mathrm{span}(v_1, \cdots, v_m)$, then either $(v_1, \cdots, v_m)$ is a basis of V or some v_i can be removed to obtain a basis of V.

Proof. Suppose $V = \mathrm{span}(v_1, \ldots, v_m)$. We start with the list $\mathcal{S} = (v_1, \ldots, v_m)$ and iteratively run through all vectors v_k for $k = 1, 2, \ldots, m$ to determine whether to keep or remove them from $\mathcal{S}$:

Step 1. If $v_1 = 0$, then remove v_1 from $\mathcal{S}$. Otherwise, leave $\mathcal{S}$ unchanged.

Step k**.** If $v_k \in \mathrm{span}(v_1, \ldots, v_{k-1})$, then remove v_k from $\mathcal{S}$. Otherwise, leave $\mathcal{S}$ unchanged.

The final list $\mathcal{S}$ still spans V since, at each step, a vector was only discarded if it was already in the span of the previous vectors. The process also ensures that no vector is in the span of the previous vectors. Hence, by the Linear Dependence Lemma 5.2.7, the final list $\mathcal{S}$ is linearly independent. It follows that $\mathcal{S}$ is a basis of V. □

Basis Extension Theorem

Every linearly independent list of vectors in a finite-dimensional vector space V can be extended to a basis of V.

Proof. Suppose V is finite-dimensional and that $(v_1, \ldots, v_m)$ is linearly independent. Since V is finite-dimensional, there exists a list $(w_1, \ldots, w_n)$ of vectors that spans V. We wish to adjoin some of the w_k to $(v_1, \ldots, v_m)$ in order to create a basis of V.

Step 1. If $w_1 \in \operatorname{span}(v_1, \ldots, v_m)$, then let $\mathcal{S} = (v_1, \ldots, v_m)$. Otherwise, $\mathcal{S} = (v_1, \ldots, v_m, w_1)$.

Step k. If $w_k \in \operatorname{span}(\mathcal{S})$, then leave $\mathcal{S}$ unchanged. Otherwise, adjoin w_k to $\mathcal{S}$.

After each step, the list $\mathcal{S}$ is still linearly independent since we only adjoined w_k if w_k was not in the span of the previous vectors. After n steps, $w_k \in \operatorname{span}(\mathcal{S})$ for all $k = 1, 2, \ldots, n$. Since $(w_1, \ldots, w_n)$ was a spanning list, $\mathcal{S}$ spans V so that $\mathcal{S}$ is indeed a basis of V. $\square$

Pb 1.2.16

Let V denote the set of all $m \times n$ matrices with real entries; so V is a vector space over $\mathbb{R}$ by Example 2. Let F be the field of rational numbers. Is V a vector space over F with the usual definitions of matrix addition and scalar multiplication?

- The sum of two matrices with real entries is a matrix with real entries
- The product of a matrix with real entries by a rational number is a matrix with real entries
- (VS 1-4) are independent of the choice of field, so they still hold when the field is Q
- (VS 5): $1 \in Q$ and for a matrix $M \in \mathsf{V}$, $1M = M$
- (VS 6-8) hold because $Q \subset R$, so elements in Q share all additive and multiplicative properties of elements in R

Thus V is a vector space.

Pb 1.3.23

Let W_1 and W_2 be subspaces of a vector space V.

a. Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

b. Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

(a) $\mathsf{W}_1 + \mathsf{W}_2$ is a subspace of W: Closed under vector addition, because if $u, v \in \mathsf{W}_1 + \mathsf{W}_2$, then there exist $u_1, v_1 \in \mathsf{W}_1$ and $u_2, v_2 \in \mathsf{W}_2$ such that $u = u_1 + u_2$ and $v = v_1 + v_2$, and then $u + v = u_1 + u_2 + v_1 + v_2 = (u_1 + v_1) + (u_2 + v_2) \in \mathsf{W}_1 + \mathsf{W}_2$. For scalar multiplication, $au = a(u_1 + u_2) = au_1 + au_2 \in \mathsf{W}_1 + \mathsf{W}_2$. Finally, $\mathsf{W}_1 + \mathsf{W}_2$ contains $\mathit{0}$ since both $\mathsf{W}_1, \mathsf{W}_2$ are subspaces and therefore contain $\mathit{0}$.

$\mathsf{W}_1 + \mathsf{W}_2$ contains both W_1 and W_2: Every vector in $\mathsf{W}_1 + \mathsf{W}_2$ has the form $x + y$ with $x \in \mathsf{W}_1$, $y \in \mathsf{W}_2$. Set $y = \mathit{0}$ to obtain all vectors in W_1 and $x = \mathit{0}$ to obtain all vectors in W_2. That is, any vector $x \in \mathsf{W}_1$ or $y \in \mathsf{W}_2$ is also present in $\mathsf{W}_1 + \mathsf{W}_2$.

(b) A subspace W of V that contains both W_1 and W_2 must also contain all vectors of the form $x + y$ with $x \in \mathsf{W}_1$, $y \in \mathsf{W}_2$, since it is closed under addition. Therefore it contains $\mathsf{W}_1 + \mathsf{W}_2$.

Pb 1.4.12

Show that a subset W of a vector space V is a subspace of V if and only if $\textbf{span}(W) = W$.

If W is a subspace of V, then it is closed under addition and scalar multiplication and contains all vectors of the form $v = a_1u_1 + \cdots a_nu_n$ with $u_1, \ldots, u_n \in$ W. But span(W) consists of linear combinations of vectors in W, so span(W) $\subset$ W. Moreover, clearly each vector in W is also in span(W), so W $\subset$ span(W). Therefore span(W) $=$ W.

If span(W) $=$ W, then all linear combinations $v = a_1u_1 + \cdots a_nu_n$ with $u_1, \ldots, u_n \in$ W are in W. Hence W is closed under addition and scalar multiplication, and it contains *0*. Therefore W is a vector space.

Pb 1.5.18

Let S be a set of nonzero polynomials in $P(F)$ such that no two have the same degree. Prove that S is linearly independent.

We argue by contradiction: Suppose that S is linearly dependent. Then there exist a finite number of distinct polynomials $p_1, p_2, \ldots, p_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1p_1 + a_2p_2 + \cdots a_np_n = 0.$$

Consider the polynomial p_j of highest degree such that $a_j \neq 0$. If its degree is k, then there is a non-vanishing x^k term in the left-hand side, since each other polynomial p_i with nonzero coefficient a_i has degree less than k. But the left-hand side has to equal the zero polynomial, a contradiction. Thus S is linearly independent.

Pb 1.6.14

Find base for the following subspaces of F^5:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_1 - a_3 - a_4 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}.$$

A basis for W_1 is $\{v_1, v_2, v_3, v_4\} = \{(1,0,1,0,0), (1,0,0,1,0), (0,1,0,0,0), (0,0,0,0,1)\}$, and the dimension of W_1 is 4. The vectors are clearly a subset of W_1, linearly independent, and they span W_1 since any vector $x \in \mathsf{W}_1$ can be written $x = (a+b, c, a, b, d) = av_1 + bv_2 + cv_3 + dv_4$.

A basis for W_2 is $\{v_1, v_2\} = \{(0,1,1,1,0), (1,0,0,0,-1)\}$, and the dimension of W_2 is 2. The vectors are clearly a subset of W_2, linearly independent, and they span W_2 since any vector $x \in \mathsf{W}_2$ can be written $x = (b, a, a, a, -b) = av_1 + bv_2$.

Pb 1.6.33(a)

a. Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V.

(a) Since $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$, we have $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$. Let $\beta_1 = \{v_1, \ldots, v_m\}$ and $\beta_2 = \{w_1, \ldots, w_n\}$. Note that the basis vectors $v_1, \ldots, v_m \in \mathsf{W}_1$ and $w_1, \ldots, w_n \in \mathsf{W}_2$ are all nonzero, since they are linearly independent. Thus $\beta_1 \cap \beta_2 = \varnothing$.

Next, consider $\beta = \beta_1 \cup \beta_2 = \{v_1, \ldots, v_m, w_1, \ldots, w_n\}$. The vectors of β are linearly independent, since if $(a_1v_1 + \cdots a_nv_n) + (b_1w_1 + \cdots + b_mw_m) = 0$, the first vector is in W_1 and the second in W_2 so $a_1v_1 + \cdots a_nv_n = b_1w_1 + \cdots + b_mw_m = 0$. Since β_1, β_2 are bases, this gives $a_1 = \cdots = a_n = 0$ and $b_1 = \cdots = b_m = 0$. The vectors of β also span V, since $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$ means that each vector $v \in \mathsf{V}$ can be written as $v = x + y$ with $x \in \mathsf{W}_1$ and $y \in \mathsf{W}_2$. But β_1, β_2 are bases for $\mathsf{W}_1, \mathsf{W}_2$, so $v = x + y = a_1v_1 + \cdots a_nv_n + b_1w_1 + \cdots + b_mw_m$ for some coefficients $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. Thus $\beta_1 \cup \beta_2$ is a basis for V.

Pb 1.6.33(b)

b. Conversely, Let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $V = W_1 \oplus W_2$.

(b) Let $\beta_1 = \{v_1, \ldots, v_m\}$ and $\beta_2 = \{w_1, \ldots, w_n\}$. Since $\beta = \beta_1 \cup \beta_2$ is a basis for V, each vector $v \in \mathsf{V}$ can be written $v = (a_1 v_1 + \cdots + a_m v_m) + (b_1 w_1 + \cdots + b_m w_m) = x + y$, where $x \in \mathsf{W}_1$ and $y \in \mathsf{W}_2$. Also, a vector $v \in \mathsf{W}_1 \cap \mathsf{W}_2$ can be written in the two ways $v = a_1 v_1 + \cdots + a_m v_m = b_1 w_1 + \cdots + b_n w_n$, but this means $a_1 v_1 + \cdots + a_m v_m + (-b_1) w_1 + \cdots + (-b_n) w_n = \mathit{0}$ and since β is a basis, we have $a_1 = \cdots = a_m = 0$ and $b_1 = \cdots = b_n = 0$. Therefore, $v = \mathit{0}$ so $\mathsf{W}_1 \cap \mathsf{W}_2 = \{\mathit{0}\}$, and $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.