Math 4377/6308 Advanced Linear Algebra

Chapter 4 Review and Solution to Problems

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Pb 4.2.23

Let $A \in M_n(F)$ be an upper triangular matrix. Show that

$$\det(A) = \prod_{i=1}^{n} A_{ii}.$$

First, note that $A_{ij} = 0$ for all $i > j$. We proceed by induction on n.

- $n = 1$: Obvious as $\det(A) = A_{11}$.
- $n - 1 \Rightarrow n$: We take the determinant by expanding along the last row of A. Let $\tilde{A}_{ij}$ be the matrix obtained from A by deleting the ith row and jth column. Then

$$\det(A) = (-1)^{n+n} A_{nn} \det(\tilde{A}_{nn}) = A_{nn} \prod_{i=1}^{n-1} A_{ii} = \prod_{i=1}^{n} A_{ii}$$

by the induction hypothesis as $\tilde{A}_{nn}$ is upper triangular.

Pb 4.3.10

Suppose $M \in M_n(F)$ is nilpotent, i.e., there is a $k \geq 0$ such that $M^k = 0$. Show M is not invertible.

Proof. By the fact that $\det(AB) = \det(A)\det(B)$ and an easy induction argument, we have

$$0 = \det(0) = \det(M^k) = \prod_{i=1}^{k} \det(M) = (\det(M))^k .$$

Taking kth roots, we have $\det(M) = 0$, so M is not invertible.

Pb 4.3.11

Suppose $M \in M_n(F)$ is skew-symmetric, i.e., $M^t = -M$. Show that if n is odd, then M is not invertible. What if n is even?

Proof. We know that

$$\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M).$$

If n is odd, then $\det(M) = -det(M)$, so $\det(M) = 0$, and M is not invertible. However, this equation tells us nothing if n is even. To fully answer this question, we must include examples of skew symmetric matrices which are invertible and non-invertible for n even. For $n = 2$, both

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are skew symmetric; the first is not invertible, and the second is invertible. Now for $n > 2$, we get examples by taking block diagonal matrices using the above examples.

Pb 4.3.15

Prove that if A, $B \in M_n(F)$ are similar, then $\det(A) = \det(B)$.

Proof. If A and B are similar, then

$$A = Q^{-1}BQ$$

for some invertible Q. Taking determinants gives

$$\det(A) = \det(Q^{-1}BQ) = \det(Q^{-1})\det(B)\det(Q)$$

$$= \frac{1}{\det(Q)}\det(B)\det(Q) = \det(B).$$

Pb 4.3.21

Suppose that $M \in M_n(F)$ can be written in the block upper triangular form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in M_k(F)$ and $C \in M_{n-k}(F)$. Prove that

$$\det(M) = \det(A)\det(C).$$

First, We proceed by induction on n.

- $n = 2$: Obvious as

$$\det(M) = \det \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = ac.$$

- $n - 1 \Rightarrow n$: We take the determinant by expanding along the first column of M. Let $\tilde{M}_{ij}$ be the matrix obtained from M by deleting the ith row and jth column.

First, note that $M_{i1} = 0$ for all $i > k$. For $i \leq k$, $M_{i1} = A_{i1}$ and

$$\det \tilde{M}_{i1} = \det \begin{pmatrix} \tilde{A}_{i1} & \tilde{B} \\ 0 & C \end{pmatrix} = \det(\tilde{A}_{i1}) \det(C)$$

by the induction hypothesis as $\tilde{M}_{i1}$ is block upper triangular. Then

$$\begin{aligned} \det(M) &= \sum_{i=1}^{n} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) = \sum_{i=1}^{k} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) \\ &= \left(\sum_{i=1}^{k} (-1)^{i+1} A_{i1} \det(\tilde{A}_{i1}) \right) \det(C) = \det(A) \det(C). \end{aligned}$$