Math 4377/6308 Advanced Linear Algebra

Chapter 5 Review and Solution to Problems

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Pb 5.1.9

Prove that the eigenvalues of an upper triangular matrix A are the diagonal entries of A.

Let A be an upper triangular matrix. Notice that λI_n is also an upper triangular matrix, thus $A - \lambda I_n$ is upper triangular. From problem 4.2.23 (which we proved on a previous homework) we know that the $\det(A - \lambda I_n)$ is the product of the diagonal entries, giving

$$p(\lambda) = \det(A - \lambda I_n) = \prod_{i=1}^{n}(a_{ii} - \lambda) = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda)$$

where a_{ii} are the diagonal entries of A. This is the characteristic polynomial of A and its roots are a_{ii} for all i. Thus the eigenvalues of A are its diagonal entries.

Pb 5.1.12

(a) Prove that similar matrices have the same characteristic polynomial.

(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

(a) Let A and B be similar, i.e., $\exists Q$ invertible such that $B = Q^{-1}AQ$. Note that $\det(Q^{-1}) = (\det(Q))^{-1}$. We have

$$\begin{aligned} p_B(\lambda) &= \det(B - \lambda I_n) = \det(Q^{-1}AQ - \lambda I_n) = \det(Q^{-1}AQ - \lambda Q^{-1} I_n Q) \\ &= \det(Q^{-1}(A - \lambda I_n)Q) = \det(Q^{-1}) \det(A - \lambda I_n) \det(Q) \\ &= \det(A - \lambda I_n)(\det(Q))^{-1} \det(Q) = \det(A - \lambda I_n) = p_A(\lambda). \end{aligned}$$

(b) Let T be a linear operator on a finite-dimensional vector space V, and let β and γ are any ordered bases for V, then $[T]_\beta$ is similar to $[T]_\gamma$. Result follows by (a).

Pb 5.1.14

For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

We know that $\det(A^t) = \det(A)$ so a simple calculation gives

$$p_A(\lambda) = \det(A - \lambda I_n) = \det((A - \lambda I_n)^t) = \det(A^t - \lambda I_n) = p_{A^t}(\lambda),$$

since λI_n is symmetric. Thus A and A^t have the same characteristic polynomial.

Pb 5.1.20

Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

Note that

$$f(t) = \det(A - tI_n) \quad \Rightarrow \quad f(0) = \det(A - 0 \cdot I_n) = \det(A).$$

Also, we have

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \quad \Rightarrow \quad f(0) = a_0.$$

Thus $a_0 = \det(A)$. From Corollary of Theorem 4.7, A is invertible if and only if $\det(A) = a_0 \neq 0$.

Pb 5.2.7

For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, find an expression for A^n, where n is an arbitrary positive integer.

Note that A has two distinct eigenvalues 5 and -1, thus is diagonalizable, i.e.,

$$Q^{-1}AQ = D \quad \text{with } D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Note that $Q^{-1} = \frac{1}{3}Q$. So we have

$$\begin{aligned} A^n = QD^nQ^{-1} &= \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 5^n + 2\cdot(-1)^n & 2\cdot 5^n + 2\cdot(-1)^{n+1} \\ 5^n + (-1)^{n+1} & 2\cdot 5^n + (-1)^n \end{pmatrix}. \end{aligned}$$

Pb 5.2.12

Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1}. Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

(a) Let $E_{T,\lambda} = N(T - \lambda I_V)$ and $E_{T^{-1},\lambda^{-1}} = N(T^{-1} - \lambda^{-1} I_V)$.
($E_{T,\lambda} \subseteq E_{T^{-1},\lambda^{-1}}$) If $x \in E_{T,\lambda}$, then $T(x) = \lambda x$. Applying T^{-1} to both sides gives

$$x = T^{-1}(T(x)) = T^{-1}(\lambda x) = \lambda T^{-1}(x) \quad \Rightarrow \quad T^{-1}(x) = \lambda^{-1} x,$$

since T is invertible, $\lambda \neq 0$.
($E_{T^{-1},\lambda^{-1}} \subseteq E_{T,\lambda}$) If $x \in E_{T^{-1},\lambda^{-1}}$, then $T^{-1}(x) = \lambda^{-1} x$. Applying T to both sides gives

$$x = T(T^{-1}(x)) = T(\lambda^{-1} x) = \lambda^{-1} T(x) \quad \Rightarrow \quad T(x) = \lambda x.$$

(b) If T is diagonalizable, then there is a basis β for V such that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Since T is invertible, $\lambda_j \neq 0$. From Theorem 2.18, we have

$$[T^{-1}]_\beta = ([T]_\beta)^{-1} = \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix}$$

Since $[T^{-1}]_\beta$ is diagonal, T^{-1} is diagonalizable.

Pb 5.2.13

Let $A \in M_{n\times n}(F)$. Recall from Exercise 14 of Section 5.1 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

(a) (a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue λ, $\dim(E_\lambda) = \dim(E'_\lambda)$.

(c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

(a) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$, and we have $E_{\lambda_1} = \mathrm{span}\{(1,0)^t\}$ and $E_{\lambda_2} = \mathrm{span}\{(1,1)^t\}$. However, we have $A^t = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $E_{\lambda_1} = \mathrm{span}\{(1,-1)^t\}$ and $E_{\lambda_2} = \mathrm{span}\{(0,1)^t\}$.

(b) Note that $E_\lambda = N(A - \lambda I)$ and $E'_\lambda = N(A^t - \lambda I)$. By the dimension theorem (Theorem 2.3), we have

$$\begin{aligned} \dim(E_\lambda) &= \dim(N(A - \lambda I)) = n - \operatorname{rank}(A - \lambda I) = n - \operatorname{rank}((A - \lambda I)^t) \\ &= n - \operatorname{rank}(A^t - \lambda I) = \dim(N(A^t - \lambda I)) = \dim(E'_\lambda) \end{aligned}$$

since $\operatorname{rank} B^t = \operatorname{rank} B$ for any matrix B.

(c) If A is diagonalizable, then from Theorem 5.6, the characteristic polynomial of A splits. Let m_λ be the multiplicity of λ as an eigenvalue of A. From Theorem 5.9, we have $\dim(E_\lambda) = m_\lambda$. Note that m_λ is also the multiplicity of λ as an eigenvalue of A'. From (a), we have $\dim(E'_\lambda) = \dim(E_\lambda) = m_\lambda$. From Theorem 5.9, A^t is diagonalizable.

Pb 5.2.18

(a) Prove that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., $UT = TU$).

(b) Prove that if A and B are simultaneously diagonalizable matrices, then A and B commute (i.e., $AB = BA$).

(a) Note that if D_1 and D_2 are diagonal matrices, then $D_1 D_2 = D_2 D_1$. If T and U are simultaneously diagonalizable operators, then there is a basis β for V such that $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Using that fact and Theorem 2.11, we get

$$[TU]_\beta = [T]_\beta [U]_\beta = [U]_\beta [T]_\beta = [UT]_\beta.$$

By Theorem 2.20, we can conclude from $[TU]_\beta = [UT]_\beta$ that $TU = UT$.

(b) If A and B are simultaneously diagonalizable matrices, then there is an invertible matrix such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal. As noted above, this means that these matrices commute. Then

$$Q^{-1}ABQ = (Q^{-1}AQ)(Q^{-1}BQ) = (Q^{-1}BQ)(Q^{-1}AQ) = Q^{-1}BAQ.$$

Multiplying the above by Q on the left and Q^{-1} on the right gives $AB = BA$.