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40 points 1. Label the following statements are true or false.

(1) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

(2) Any set containing the zero vector is linearly dependent.

(3) Subsets of linearly dependent sets are linearly dependent.

(4) Subsets of linearly independent sets are linearly independent.

(5) Every vector space that is generated by a finite set has a basis.

(6) Every vector space has a finite basis.

(7) If a vector space has a finite basis, then the number of vectors in every basis is the same.

(8) The dimension of $P_n(F)$ is n.

(9) The dimension of $M_{m\times n}(F)$ is $m+n$.

(10) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.

(11) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

(12) If $T : V \to W$ is linear, then T preserves sums and scalar products.

(13) If $T : V \to W$ is linear, then $\text{nullity}(T) + \text{rank}(T) = \dim(W)$.

(14) If $T : V \to W$ is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.

(15) If $T, U : V \to W$ are both linear and agree on a basis for V, then $T = U$.

(16) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

(17) Any homogeneous system of linear equations has at least one solution.

(18) A matrix $A \in M_{n\times n}(F)$ has rank n if and only if $\det(A) \neq 0$.

(19) Similar matrices always have the same eigenvalues.

(20) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

20 points 2. The first four Chebyshev polynomials are 1, x, $2x^2-1$, and $4x^3-3x$. These polynomials arise naturally in the study of certain important differential equations. Show that the first four Chebyshev polynomials form a basis of $P_3(\mathbb{R})$.

20 points 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(a, b, c) = (3a, -2a + c, b).$$

Prove that T is an isomorphism and find T^{-1}.

45 points 4. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(a, b, c) = (a - b, 2c)$.

(a) Show that T is a linear transformation.

(b) Find bases for the null space and the range of T.

(c) Compute the nullity and rank of T, and verify the dimension theorem.

45 points 5. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

(c) Suppose

$$W_1 = \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}, \quad W_2 = \text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_q\}.$$

where $\mathbf{u}_1, \cdots, \mathbf{u}_p$ and $\mathbf{v}_1, \cdots, \mathbf{v}_q$ are vectors in V. Show that

$$W_1 + W_2 = \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}.$$

30 points 6. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W.

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

30 points 7. Find the inverse of each of the following elementary matrices

$$\text{(a)} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \text{(b)} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{(c)} \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

30 points 8. Let

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{pmatrix}.$$

Express A^{-1} as a product of elementary matrices.

40 points 9. Compute the determinant of each of the following matrices

$$\text{(a)} \quad \begin{pmatrix} 2 & 9 & 7 & 11 \\ 2 & 7 & 6 & 10 \\ 0 & 5 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \text{(b)} \quad \begin{pmatrix} 0 & 0 & 3 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 1 & 5 \\ 1 & 0 & 3 & 4 \end{pmatrix}, \quad \text{(c)} \quad \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & -3 & 5 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \end{pmatrix}, \quad \text{(d)} \quad \begin{pmatrix} 0 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

30 points 10. For $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$, find an expression for e^A.

30 points 11. Suppose that $M \in M_n(F)$ can be written in the block upper triangular form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in M_k(F)$ and $C \in M_{n-k}(F)$. Prove that

$$\det(M) = \det(A)\det(C).$$

40 points 12. Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1}. Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

30 points 13. (**BONUS PROBLEM**) Let A be an $m \times n$ matrix with rank m and B be an $n \times p$ matrix with rank n. Determine the rank of AB. Justify your answer.

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Problem 1.

(1) fasle, (2) true, (3) fasle, (4) true, (5) true, (6) fasle, (7) true, (8) fasle, (9) fasle, (10) true, (11) true, (12) true, (13) false, (14) fasle, (15) true, (16) true, (17) true, (18) true, (19) true, (20) false

Problem 2.
Let $\beta = \{1, x, 2x^2 - 1, 4x^3 - 3x\}$ and let $\gamma = \{1, x, x^2, x^3\}$ be the standard ordered basis for $P_3(\mathbb{R})$. We have the coordinate vectors of β in γ as:

$$[1]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad [x]_\gamma = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad [-1+2x^2]_\gamma = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \quad [-3x+4x^3]_\gamma = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 4 \end{pmatrix}$$

Note that the matrix with the coordinate vectors as columns have four pivots

$$Q = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Then $\{[1]_\gamma, [x]_\gamma, [-1+2x^2]_\gamma, [-3x+4x^3]_\gamma\}$ is linearly independent. By Thorem 2.21, β is linearly independent. Combined with the fact that $|\beta| = \dim(P_3(\mathbb{R})) = 4$, β is a basis for $P_3(\mathbb{R})$.

Problem 3. Let $\beta = \{e_1, e_2, e_3\}$ be the standard ordered basis for R^3. The matrix representation of T in β is

$$[T]_\beta = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Note that the augmented matrix

$$[[T]_\beta \,|\, I_3] = \begin{pmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2/3 & 1 & 0 \end{pmatrix} = \left[I_3 \,|\, ([T]_\beta)^{-1}\right]$$

So we have

$$[T]_\beta^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 1 \\ 2/3 & 1 & 0 \end{pmatrix}$$

By Theorem 2.18, T is invertible and $[T^{-1}]_\beta = ([T]_\beta)^{-1}$. We have

$$T^{-1}(a, b, c) = (a/3, c, 2a/3 + b).$$

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Problem 4.

(a) Note that, in the matrix and column vector notation, we have

$$x \in \mathbb{R}^3 \mapsto T(x) = Ax, \quad A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then, T is linear.

(b) To find the null space of T, row reduce the augmented matrix corresponding to $Ax = 0$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Then

$$\text{the null space of } T = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

The range of T is the column space of A and we have

$$\text{the range of } T = \text{span}\,\{\text{pivot columns of } A\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

(c) The nullity of T is 1 and the rank of T is 2 We have

$$\text{nullity}(T) + \text{rank}(T) = 1 + 2 = 3 = \dim(\mathbb{R}^3).$$

Then, the dimension theorem is verified.

Problem 5.

(a) $W_1 + W_2$ is a subspace of W: Closed under vector addition, because if u, $v \in W_1 + W_2$, then there exist u_1, $v_1 \in W_1$ and u_2, $v_2 \in W_2$ such that $u = u_1 + u_2$ and $v = v_1 + v_2$, and then $u + v = u_1 + u_2 + v_1 + v_2 = (u_1 + v_1) + (u_2 + v_2) \in W_1 + W_2$. For scalar multiplication, $au = a(u_1 + u_2) = au_1 + au_2 \in W_1 + W_2$. Finally, $W_1 + W_2$ contains 0 since both W_1, W_2 are subspaces and therefore contain 0. $W_1 + W_2$ contains both W_1 and W_2: Every vector in $W_1 + W_2$ has the form $x + y$ with $x \in W_1$, $y \in W_2$. Set $y = 0$ to obtain all vectors in W_1 and $x = 0$ to obtain all vectors in W_2. That is, any vector $x \in W_1$ or $y \in W_2$ is also present in $W_1 + W_2$.

(b) A subspace W of V that contains both W_1 and W_2 must also contain all vectors of the form $x + y$ with $x \in W_1$, $y \in W_2$, since it is closed under addition. Therefore it contains $W_1 + W_2$.

(c) ($\Rightarrow$) $W_1 + W_2 \subseteq \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$: For any $\mathbf{u} \in W_1 = \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ and $\mathbf{v} \in W_2 = \text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_q\}$, there exist $c_1, \cdots, c_p$ and $d_1, \cdots, d_q$ such that

$$\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p, \quad \mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q.$$

Then

$$\mathbf{u} + \mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p + d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q \in \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$$

($\Leftarrow$) $\text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\} \subseteq W_1 + W_2$: For any $\mathbf{w} \in \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$, there exist $c_1, \cdots, c_p$ and $d_1, \cdots, d_q$ such that

$$\mathbf{w} = (c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) + (d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q) \in W_1 + W_2$$

Problem 6.

(a) (1) $T(V_0)$ contains 0_W, since $0_V \in V_0$ and $T(0_V) = 0_W$. (2) Let $u_1, u_2 \in T(V_0)$, then there exist $v_1, v_2 \in V_0$ such that $T(v_1) = u_1$ and $T(v_2) = u_2$. Then $v_1 + v_2 \in V_0$, and $T(V_0) \ni T(v_1 + v_2) = T(v_1) + T(v_2) = u_1 + u_2$. (3) Similarly for scalar multiplication, let $u \in T(V_0)$, then there exists $v \in V_0$ such that $T(v) = u$. Then $av \in V_0$, and $T(V_0) \ni T(av) = aT(v) = au$. Combining (1)-(3) shows that $T(V_0)$ is a subspace of W.

(b) Let $\beta = \{u_1, \cdots, u_n\}$ be a basis for V_0. $T(\beta)$ is then a basis for $T(V_0)$, since it spans $T(V_0)$ and its vectors are linearly independent:

$$a_1T(u_1) + \cdots + a_nT(u_n) = T(a_1u_1 + \cdots + a_nu_n) = 0$$

gives $a_1u_1 + \cdots + a_nu_n = 0$ since T is an isomorphism, and $a_1 = \cdots = a_n = 0$ since β is a basis for V_0. Thus, $n = \dim(V_0) = \dim(T(V_0))$.

Problem 7.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

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Problem 8.

Perform the row operations to reduce the matrix A to the identity matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 0 & -3 & 13 \end{bmatrix} \quad \text{with} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -4 \\ 0 & -3 & 13 \end{bmatrix} \quad \text{with} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad E_5 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad \text{with} \quad E_6 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In matrix form, we have

$$E_6(E_5(E_4(E_3(E_2(E_1A))))) = I_3.$$

Therefore, by the uniqueness of the inverse matrix of A, we have

$$A^{-1} = E_6E_5E_4E_3E_2E_1.$$

Problem 9.

(a)

$$\det\begin{pmatrix} 2 & 9 & 7 & 11 \\ 2 & 7 & 6 & 10 \\ 0 & 5 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (-5)\cdot\det\begin{pmatrix} 2 & 7 & 11 \\ 2 & 6 & 10 \\ 1 & 3 & 4 \end{pmatrix} = (-5)\cdot\det\begin{pmatrix} 2 & 7 & 11 \\ 0 & 0 & 2 \\ 1 & 3 & 4 \end{pmatrix} = (-5)\cdot(-2)\cdot\det\begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$$

$$= (-5)\cdot(-2)\cdot(2\cdot 3 - 1\cdot 7) = -10.$$

(b)

$$\det\begin{pmatrix} 0 & 0 & 3 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 1 & 5 \\ 1 & 0 & 3 & 4 \end{pmatrix} = (-2)\cdot\det\begin{pmatrix} 0 & 3 & 5 \\ 0 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix} = (-2)\cdot 1\cdot\det\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} = (-2)\cdot 1\cdot(3\cdot 1 - 2\cdot 5) = 14.$$

(c)

$$\det\begin{pmatrix}0&0&0&4\\0&0&-3&5\\0&1&2&3\\2&3&4&5\end{pmatrix} = (-2)\cdot\det\begin{pmatrix}0&0&4\\0&-3&5\\1&2&3\end{pmatrix} = (-2)\cdot 1\cdot\det\begin{pmatrix}0&4\\-3&5\end{pmatrix}$$
$$= (-2)\cdot 1\cdot(-(-3))\cdot 4 = -24.$$

(d)

$$\det\begin{pmatrix}0&0&0&4\\2&0&0&0\\0&1&0&0\\0&0&3&0\end{pmatrix} = (-2)\cdot\det\begin{pmatrix}0&0&4\\1&0&0\\0&3&0\end{pmatrix} = (-2)\cdot(-1)\cdot\det\begin{pmatrix}0&4\\3&0\end{pmatrix}$$
$$= (-2)\cdot(-1)\cdot(-3)\cdot 4 = -24.$$

Problem 10.
Note that the characteristic polynomial of A is

$$p_A(\lambda) = \det(A-\lambda I_2) = \lambda^2 - \operatorname{trace}(A)\lambda + \det(A) = \lambda^2 - 6\lambda + 5 = (\lambda-5)(\lambda-1).$$

Then A has two distinct eigenvalues 5 and 1, thus is diagonalizable. Note that $(1,3)^t$ is an eigenvector corresponding to the eigenvalue 5 and $(1,-1)^t$ an eigenvector corresponding to the eigenvalue 1. We have

$$Q^{-1}AQ = D \quad \text{with } D = \begin{pmatrix}5&0\\0&1\end{pmatrix}, \quad Q = \begin{pmatrix}1&1\\3&-1\end{pmatrix}.$$

Note that $Q^{-1} = \frac{1}{4}\begin{pmatrix}1&1\\3&-1\end{pmatrix}$. So we have

$$A^n = (QDQ^{-1})\cdots(QDQ^{-1}) = QD^nQ^{-1} = \frac{1}{4}\begin{pmatrix}1&1\\3&-1\end{pmatrix}\begin{pmatrix}e^5&0\\0&e\end{pmatrix}\begin{pmatrix}1&1\\3&-1\end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix}e^5+3e & e^5-e\\ 3e^5-3e & 3e^5+e\end{pmatrix}.$$

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Problem 11.

We proceed by induction on n.

- $n = 2$: Obvious as
$$\det(M) = \det\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = ac.$$

- $n-1 \Rightarrow n$: We take the determinant by expanding along the first column of M. Let $\tilde{M}_{ij}$ be the matrix obtained from M by deleting the ith row and jth column. First, note that $M_{i1} = 0$ for all $i > k$. For $i \leq k$, $M_{i1} = A_{i1}$ and
$$\det \tilde{M}_{i1} = \det\begin{pmatrix} \tilde{A}_{i1} & \tilde{B} \\ 0 & C \end{pmatrix} = \det(\tilde{A}_{i1})\det(C)$$
by the induction hypothesis as $\tilde{M}_{i1}$ is block upper triangular. Then
$$\begin{aligned}\det(M) &= \sum_{i=1}^{n}(-1)^{i+1}M_{i1}\det(\tilde{M}_{i1}) = \sum_{i=1}^{k}(-1)^{i+1}M_{i1}\det(\tilde{M}_{i1}) \\ &= \left(\sum_{i=1}^{k}(-1)^{i+1}A_{i1}\det(\tilde{A}_{i1})\right)\det(C) = \det(A)\det(C).\end{aligned}$$

Problem 12.

(a) Let $E_{T,\lambda} = N(T - \lambda I_V)$ and $E_{T^{-1},\lambda^{-1}} = N(T^{-1} - \lambda^{-1}I_V)$.
($E_{T,\lambda} \subseteq E_{T^{-1},\lambda^{-1}}$) If $x \in E_{T,\lambda}$, then $T(x) = \lambda x$. Applying T^{-1} to both sides gives
$$x = T^{-1}(T(x)) = T^{-1}(\lambda x) = \lambda T^{-1}(x) \quad \Rightarrow \quad T^{-1}(x) = \lambda^{-1}x,$$
since T is invertible, $\lambda \neq 0$.
($E_{T^{-1},\lambda^{-1}} \subseteq E_{T,\lambda}$) If $x \in E_{T^{-1},\lambda^{-1}}$, then $T^{-1}(x) = \lambda^{-1}x$. Applying T to both sides gives
$$x = T(T^{-1}(x)) = T(\lambda^{-1}x) = \lambda^{-1}T(x) \quad \Rightarrow \quad T(x) = \lambda x.$$

(b) If T is diagonalizable, then there is a basis β for V such that
$$[T]_\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$
Since T is invertible, $\lambda_j \neq 0$. From Theorem 2.18, we have
$$[T^{-1}]_\beta = ([T]_\beta)^{-1} = \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix}$$
Since $[T^{-1}]_\beta$ is diagonal, T^{-1} is diagonalizable.

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Problem 13. (BONUS PROBLEM)
If A is $m \times n$ with $\text{rank}(A) = m$, then $L_A : F^n \to F^m$ with

$$R(A) = R(L_A) = L_A(F^n) = F^m.$$

If B is $n \times p$ with $\text{rank}(B) = n$, then $L_B : F^p \to F^n$ with

$$R(B) = R(L_B) = L_B(F^p) = F^n.$$

Note that AB is $m \times p$ and $L_{AB} : F^p \to F^m$. Therefore,

$$R(AB) = R(L_{AB}) = L_{AB}(F^p) = L_A(L_B(F^p)) = L_A(F^n) = F^m$$

and

$$\text{rank}(AB) = m.$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.