Name and ID: ______________________

20 points 1. Label the following statements are true or false

(a) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

(b) An $n \times n$ matrix having rank n is invertible.

(c) Any homogeneous system of linear equations has at least one solution.

(d) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.

(e) The determinant of an upper triangular $n \times n$ matrix is the product of its diagonal entries.

(f) A matrix $A \in M_{n\times n}(F)$ has rank n if and only if $\det(A) \neq 0$.

(g) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.

(h) Similar matrices always have the same eigenvalues.

(i) Let $A \in M_{n\times n}(F)$ and $\beta = \{v_1, v_2, \cdots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose jth column is v_j $(1 \leq j \leq n)$, then $Q^{-1}AQ$ is a diagonal matrix.

(j) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the characteristic polynomial of T splits and the multiplicity of each eigenvalue λ equals the dimension of E_λ.

10 points 2. Describe the solution set of

$$2x_1 + 4x_2 - 6x_3 = 0, \quad 4x_1 + 8x_2 - 10x_3 = 0;$$

compare it to the solution set

$$2x_1 + 4x_2 - 6x_3 = 0, \quad 4x_1 + 8x_2 - 10x_3 = 4.$$

15 points 3. Prove that if B is a $n \times 1$ matrix and C is a $1 \times n$ matrix, then the $n \times n$ matrix BC has rank at most 1. Conversely, show that if A is any $n \times n$ matrix having rank 1, then there exists a $n \times 1$ matrix B and a $1 \times n$ matrix C such that $A = BC$.

10 points 4. Compute the determinant of each of the following matrices

$$\text{(a)} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 4 & 2 & 1 & 3 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 5 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

20 points 5. Let $A \in M_{n\times n}(F)$ such that $A^k = 0$ for some positive integer k. Prove that A is not invertible.

15 points 6. For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, find an expression for e^A. (Recall that the exponential of A, denoted by e^A, is the matrix given by the power series $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$).

15 points 7. Prove that similar matrices have the same characteristic polynomial.

20 points 8. (**BONUS PROBLEM**) Let $A \in M_{n\times n}(F)$. Prove that if A is diagonalizable, then A^t is also diagonalizable.

Name and ID: ______________________________

Problem 1.

(a) True. From Corollary 2 in Page 158, $\text{rank}(A) = \dim(\text{row space of } A) = \dim(\text{column space of } A)$.

(b) True. From Theorem 2.5, $L_A : F^n \to F^n$ is invertible if and only if $\text{rank}(L_A) = \dim(F^n)$, i.e., $\text{rank}(A) = n$. From Corollary 2 in Page 102, A is invertible if and only if L_A is invertible. Then A is invertible if and only if $\text{rank}(A) = n$.

(c) True. Any homogeneous system of linear equations has zero as a solution.

(d) False. For example, the system that $0x = 1$ has no solution while the corresponding homogeneous system $0x = 0$ has a solution.

(e) True. It is from Problem 4.2.23 in Page 222.

(f) True. From Property 7 in Page 236, A is invertible if and only if $\det(A) \neq 0$. Combined with the result (b) above, A has rank n if and only if $\det(A) \neq 0$.

(g) True. If $v \in E_\lambda \setminus \{0\}$, i.e., v is an eigenvector of A, then $cv \in E_\lambda \setminus \{0\}$ for any $c \in \mathbb{R} \setminus \{0\}$.

(h) True. From Problem 5.1.12(a), similar matrices have the same charateristic polynomial, thus have the same eigenvalues.

(i) True. Let $Q = (v_1, \cdots, v_n)$ be the $n \times n$ matrix whose jth column is v_j $(1 \leq j \leq n)$ such that $Av_j = \lambda_j v_j$, then $AQ = A(v_1, \cdots, v_n) = (Av_1, \cdots, Av_n) = (\lambda_1 v_1, \cdots, \lambda_n v_n) = (v_1, \cdots, v_n)\text{diag}(\lambda_1, \cdots, \lambda_n) = QD$ with $D = \text{diag}(\lambda_1, \cdots, \lambda_n)$. Therefore, $Q^{-1}AQ = D$ is a diagonal matrix.

(j) True. It is from Theorem 5.9 (i.e., the test for diagonalization).

Problem 2.

Note that the corresponding augmented matrix to

$$2x_1 + 4x_2 - 6x_3 = 0, \quad 4x_1 + 8x_2 - 10x_3 = 0.$$

is

$$\begin{pmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -10 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The vector form of the solution is

$$x = \begin{pmatrix} -2x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

The corresponding augmented matrix to

$$2x_1 + 4x_2 - 6x_3 = 0, \quad 4x_1 + 8x_2 - 10x_3 = 4.$$

is

$$\begin{pmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

The vector form of the solution is

$$x = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

Name and ID: ______________________________

Problem 3.

($\Rightarrow$) If $B = b \in F^{n\times 1}$ and $C = c^T = (c_1, \cdots, c_n) \in F^{1\times n}$, then

$$BC = bc^T = b(c_1, \cdots, c_n) = (c_1 b, \cdots, c_n b),$$

thus $\text{range}(BC) = \text{span}(\{b\})$ and $\text{rank}(BC) = \dim(\text{span}(\{b\})) \le 1$.

($\Leftarrow$) If $A = (a_1, \cdots, a_n) \in F^{n\times n}$ have rank 1, then

$$\text{range}(A) = \text{span}(\{a_1, \cdots, a_n\}) = \text{span}(\{b\})$$

for some $b \neq 0 \in F^{n\times 1}$, and $\exists c^T = (c_1, \cdots, c_n) \in F^{1\times n}$ such that

$$a_1 = c_1 b, \cdots, a_n = c_n b.$$

Therefore,

$$A = (c_1 b, \cdots, c_n b) = b(c_1, \cdots, c_n) = BC$$

with $B = b$ and $C = c^T$.

Problem 4.

(a)

$$\det\begin{pmatrix}1 & 0 & 2 & 3\\ 4 & 2 & 1 & 3\\ 0 & 0 & 3 & 4\\ 0 & 0 & 4 & 5\end{pmatrix} = \det\begin{pmatrix}1 & 0\\ 4 & 2\end{pmatrix}\det\begin{pmatrix}3 & 4\\ 4 & 5\end{pmatrix} = (1\cdot 2)(3\cdot 5 - 4\cdot 4) = -2.$$

(b)

$$\det\begin{pmatrix}1 & 2 & 3 & 4\\ 0 & 2 & 3 & 4\\ 0 & 0 & 3 & 4\\ 0 & 0 & 0 & 4\end{pmatrix} = 1\cdot 2\cdot 3\cdot 4 = 24.$$

(c)

$$\det\begin{pmatrix}0 & 0 & 2 & 0\\ 0 & 1 & 0 & 0\\ 3 & 0 & 0 & 0\\ 0 & 0 & 0 & 4\end{pmatrix} = -\det\begin{pmatrix}3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 4\end{pmatrix} = -(3\cdot 1\cdot 2\cdot 4) = -24.$$

Name and ID: ____________________

Problem 5.

Proof. By the fact that $\det(BC) = \det(B)\det(C)$ and an easy induction argument, we have

$$0 = \det(0) = \det(A^k) = \prod_{i=1}^{k} \det(A) = (\det(A))^k .$$

Taking kth roots, we have $\det(A) = 0$, so A is not invertible.

Problem 6.

Note that A has two distinct eigenvalues 5 and -1, thus is diagonalizable, i.e.,

$$Q^{-1}AQ = D \quad \text{with } D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad Q^{-1} = \frac{1}{3}Q.$$

Note that

$$A^k = QD^kQ^{-1}, \quad e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k = Q\left(\sum_{k=0}^{\infty} \frac{1}{k!}D^k\right)Q^{-1} = Qe^DQ^{-1}$$

So we have

$$\begin{aligned} e^A = Qe^DQ^{-1} &= \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^5 & 0 \\ 0 & e^{-1} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} e^5 + 2e^{-1} & 2e^5 - 2e^{-1} \\ e^5 - e^{-1} & 2e^5 + e^{-1} \end{pmatrix}. \end{aligned}$$

Name and ID: ______________________________

Problem 7.

Let A and B be similar, i.e., $\exists Q$ invertible such that $B = Q^{-1}AQ$. Note that $\det(Q^{-1}) = (\det(Q))^{-1}$. We have

$$\begin{aligned} p_B(\lambda) &= \det(B - \lambda I_n) = \det(Q^{-1}AQ - \lambda I_n) = \det(Q^{-1}AQ - \lambda Q^{-1}I_nQ) \\ &= \det(Q^{-1}(A - \lambda I_n)Q) = \det(Q^{-1})\det(A - \lambda I_n)\det(Q) \\ &= \det(A - \lambda I_n)(\det(Q))^{-1}\det(Q) = \det(A - \lambda I_n) = p_A(\lambda). \end{aligned}$$

Problem 8. (BONUS PROBLEM)

(1st proof:)

Note that $(BC)^t = C^tB^t$ for any dimension consistent matrices B and C. If A is diagonalizable, then there are an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$. Taking the transpose on both sides gives

$$D = D^t = \left(Q^{-1}AQ\right)^t = (Q)^t(A)^t\left(Q^{-1}\right)^t = \tilde{Q}^{-1}A^t\tilde{Q}$$

with $\tilde{Q} = (Q^{-1})^t$ and we have $\tilde{Q}^{-1} = Q^t$. Therefore, A^t is diagonalizable.

(2nd proof:)

If A is diagonalizable, then from Theorem 5.6, the characteristic polynomial of A splits. Let m_λ be the multiplicity of λ as an eigenvalue of A. From Theorem 5.9, we have $\dim(E_\lambda) = m_\lambda$. Note that A and A' have the the characteristic polynomial. Then m_λ is also the multiplicity of λ as an eigenvalue of A'. Note that $E_\lambda = N(A - \lambda I)$ and $E'_\lambda = N(A^t - \lambda I)$. By the dimension theorem (Theorem 2.3), we have

$$\begin{aligned} \dim(E_\lambda) &= \dim(N(A - \lambda I)) = n - \operatorname{rank}(A - \lambda I) = n - \operatorname{rank}((A - \lambda I)^t) \\ &= n - \operatorname{rank}(A^t - \lambda I) = \dim(N(A^t - \lambda I)) = \dim(E'_\lambda) \end{aligned}$$

since $\operatorname{rank}B^t = \operatorname{rank}B$ for any matrix B. Therefore, we have $\dim(E'_\lambda) = \dim(E_\lambda) = m_\lambda$. From Theorem 5.9, A^t is diagonalizable.

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.