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10 points 1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case).

(1) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

(2) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

(3) The sum of two eigenvectors of a linear operator T is always an eigenvector of T.

(4) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(5) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

(6) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

(7) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

(8) Let $A \in M_{n\times n}(F)$ and $\beta = \{v_1, v_2, \cdots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose jth column is v_j $(1 \le j \le n)$, then $Q^{-1}AQ$ is a diagonal matrix.

(9) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

(10) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

10 points 2. Prove that similar matrices have the same characteristic polynomial and hence the same eigenvalues.

10 points 3. Prove that the eigenvalues of an upper triangular matrix A are the diagonal entries of A.

10 points 4. For $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$, find an expression for A^n, where n is an arbitrary positive integer.

10 points 5. (**BONUS PROBLEM**) Let $A \in M_{n\times n}(F)$ be invertible. Prove that if A is diagonalizable, then A^{-1} is also diagonalizable.

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Problem 1.

(1) False. For example, the identity mapping I has only one (distinct) eigenvalue 1 (with multiplicity n).

(2) False. For example, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ has two eigenvalues 1 and 2, the sum 3 is not an eigenvalue of the same matrix.

(3) False. For example, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ has two eigenvectors $(1,0)^t$ and $(0,1)^t$, the sum $(1,1)^t$ is not an eigenvector of the same matrix.

(4) False. For example, the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has only one (distinct) eigenvalue but it is diagonalizable.

(5) False. For example, the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has two linearly independent eigenvectors $(1,0)^t$ and $(0,1)^t$ corresponding to the same eigenvalue 1.

(6) False. The zero vector 0 (in E_λ) is not an eigenvector

(7) True. If $v \in E_{\lambda_1} \cap E_{\lambda_2}$ with $\lambda_1 \neq \lambda_2$, then $T(v) = \lambda_1 v = \lambda_2 v$, thus $(\lambda_1 - \lambda_2)v = 0$, implying $v = 0$.

(8) True. Let $Q = (v_1, \cdots, v_n)$ be the $n \times n$ matrix whose jth column is v_j $(1 \le j \le n)$ such that $Av_j = \lambda_j v_j$, then $AQ = A(v_1, \cdots, v_n) = (Av_1, \cdots, Av_n) = (\lambda_1 v_1, \cdots, \lambda_n v_n) = (v_1, \cdots, v_n)\text{diag}(\lambda_1, \cdots, \lambda_n) = QD$ with $D = \text{diag}(\lambda_1, \cdots, \lambda_n)$. Therefore, $Q^{-1}AQ = D$ is a diagonal matrix.

(9) False. The test for diagonalization requires that the characteristic polynomial of T splits.

(10) True. Let T be a diagonalization linear operator on a nonzero vector space. The characteristic polynomial of T has a degree greater than or equal to one and splits, thus has at least one root. Hence T has at least one eigenvalue.

Problem 2.

Let A and B be similar, i.e., $\exists Q$ invertible such that $B = Q^{-1}AQ$. Note that $\det(Q^{-1}) = (\det(Q))^{-1}$. We have

$$\begin{aligned} p_B(\lambda) &= \det(B - \lambda I_n) = \det(Q^{-1}AQ - \lambda I_n) = \det(Q^{-1}AQ - \lambda Q^{-1} I_n Q) \\ &= \det(Q^{-1}(A - \lambda I_n)Q) = \det(Q^{-1}) \det(A - \lambda I_n) \det(Q) \\ &= \det(A - \lambda I_n)(\det(Q))^{-1} \det(Q) = \det(A - \lambda I_n) = p_A(\lambda). \end{aligned}$$

Thus A and B have the same characteristic polynomial (and hence the same eigenvalues).

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Problem 3.

Let A be an upper triangular matrix. Notice that λI_n is also an upper triangular matrix, thus $A - \lambda I_n$ is upper triangular. From problem 4.2.23 (which we proved on a previous homework) we know that $\det(A - \lambda I_n)$, the determinant of an upper triangular matrix, is the product of the diagonal entries, giving

$$p(\lambda) = \det(A - \lambda I_n) = \prod_{i=1}^{n} (a_{ii} - \lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$$

where a_{ii} are the diagonal entries of A. This is the characteristic polynomial of A and its roots are a_{ii} for all i. Thus the eigenvalues of A are its diagonal entries.

Problem 4.

Note that the characteristic polynomial of A is

$$p_A(\lambda) = \det(A - \lambda I_2) = \lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).$$

Then A has two distinct eigenvalues 5 and -1, thus is diagonalizable. Note that $(1, 1)^t$ is an eigenvector corresponding to the eigenvalue 5 and $(-1, 2)^5$ an eigenvector corresponding to the eigenvalue -1. We have

$$Q^{-1}AQ = D \quad \text{with } D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}.$$

Note that $Q^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$. So we have

$$\begin{aligned} A^n = QD^nQ^{-1} &= \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2 \cdot 5^n + (-1)^n & 5^n + (-1)^{n+1} \\ 2 \cdot 5^n + 2 \cdot (-1)^{n+1} & 5^n + 2 \cdot (-1)^n \end{pmatrix}. \end{aligned}$$

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Problem 5. (BONUS PROBLEM)

Note that $(BC)^{-1} = C^{-1}B^{-1}$ for any invertible matrices B and C. If A is diagonalizable, then there is an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$. Taking the inverse on both sides gives

$$D^{-1} = \left(Q^{-1}AQ\right)^{-1} = (Q)^{-1}(A)^{-1}\left(Q^{-1}\right)^{-1} = Q^{-1}A^{-1}Q$$

since $\left(Q^{-1}\right)^{-1} = Q$. Therefore, A^{-1} is diagonalizable.

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.