Math 4377/6308 Advanced Linear Algebra

1.3 Subspaces

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1.3 Subspaces

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Subspaces

Vector spaces may be formed from subsets of other vectors spaces. These are called *subspaces*.

Definition (Subspace)

A subset W of a vector space V is called a **subspace** of V if W is a vector space in its own right under the operations obtained by restricting the operations of V to W.

Example

Note that V and $\{\mathbf{0}\}$ are subspaces of any vector space V. $\{\mathbf{0}\}$ is called the **zero subspace** of V. We call these the **trivial subspaces** of V.

Verification of Subspaces

It is clear that properties (VS 1,2,5-8) hold for any subset of vectors in a vector space. Therefore, a subset W of a vector space V is a subspace of V if and only if:

1. $\mathbf{x}+\mathbf{y} \in W$ whenever $\mathbf{x}, \mathbf{y} \in W$.
2. $c\mathbf{x} \in W$ whenever $c \in F$ and $\mathbf{x} \in W$
3. W has a zero vector
4. Each vector in W has an additive inverse in W.

Furthermore, the zero vector of W must be the same as of V, and property 4 follows from property 2 and Theorem 1.2.

Verification of Subspaces (cont.)

Theorem (1.3)

A subset W of a vector space V is a subspace of V if and only if

(a) $\mathbf{0} \in W$

(b) $\mathbf{x} + \mathbf{y} \in W$ *whenever* $\mathbf{x}, \mathbf{y} \in W$
(W is closed under vector addition)

(c) $c\mathbf{x} \in W$ *whenever* $c \in F$ *and* $\mathbf{x} \in W$
(W is closed under scalar multiplication).

A nonempty subset W of a vector space V is a subspace of V if and only if W is closed under addition and scalar multiplication or, equivalently, W is closed under linear combinations, that is,

$$\forall a, b \in F, \forall \mathbf{u}, \mathbf{v} \in W \quad \Rightarrow \quad a\mathbf{u} + b\mathbf{v} \in W$$

Subspaces: Example

Example

Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$. Show that H is a subspace of $\mathbf{R}^3$.

Solution: Verify properties a, b and c of the definition of a subspace.

a. The zero vector of $\mathbf{R}^3$ is in H (let $a =$ _______ and $b =$ _______).

b. Adding two vectors in H always produces another vector whose second entry is ______ and therefore the sum of two vectors in H is also in H. (H is closed under addition)

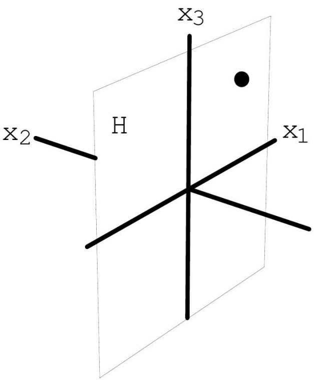
c. Multiplying a vector in H by a scalar produces another vector in H (H is closed under scalar multiplication).

Since properties a, b, and c hold, V is a subspace of $\mathbf{R}^3$.

Subspaces: Example (cont.)

Note

Vectors $(a, 0, b)$ in H look and act like the points (a, b) in $\mathbf{R}^2$.



Graphical Depiction of H

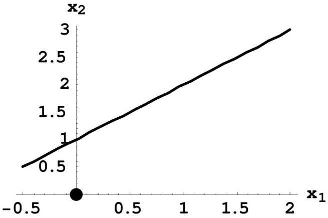
Subspaces: Example

Example

Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \text{ is real} \right\}$ a subspace of _______?

I.e., does H satisfy properties a, b and c?

Solution: For *H to be a subspace of* $\mathbf{R}^2$, *all three properties must hold*



Property (a) fails

Property (a) is not true because _________________________________.
Therefore H is not a subspace of $\mathbf{R}^2$.

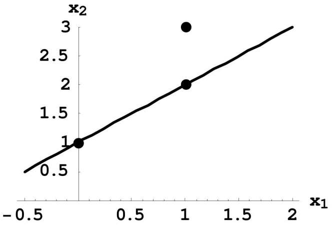
Subspaces: Example (cont.)

Another way to show that H is not a subspace of $\mathbf{R}^2$:

Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ then } \mathbf{u} + \mathbf{v} = \begin{bmatrix} \quad \end{bmatrix}$$

and so $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which is ____ in H. So property (b) fails and so H is not a subspace of $\mathbf{R}^2$.



Property (b) fails

Subspaces: Example

Example

Is $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - 5y + 7z = 0 \right\}$ a subspace of $\mathbb{R}^3$?

Subspaces: Example (Zero Vector)

Example

Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 3x - 5y = 12 \right\}$ a subspace of $\mathbb{R}^2$?
(zero vector?)

Subspaces: Example (Additive Closure)

Example

Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy = 0 \right\}$ a subspace of $\mathbb{R}^2$?
(additive closure?)

Subspaces: Example (scalar multiplication closure)

Example

Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \in \mathbb{Z}, y \in \mathbb{Z} \right\}$ a subspace of $\mathbb{R}^2$?
(scalar multiplication closure?)

Subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

Example

- The subspaces of $\mathbb{R}^2$ consist of $\{\mathbf{0}\}$, all lines through the origin, and $\mathbb{R}^2$ itself.
- The subspaces of $\mathbb{R}^3$ are $\{\mathbf{0}\}$, all lines through the origin, all planes through the origin, and $\mathbb{R}^3$.

In fact, these exhaust all subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. To prove this, we will need further tools such as the notion of bases and dimensions to be discussed soon. In particular, this shows that lines and planes that do not pass through the origin are not subspaces (which is not so hard to show!).

Intersections of Subspaces

Theorem (1.4)

Any intersection of subspaces of a vector space V is a subspace of V.

Unions of Subspaces

However, the union of subspaces is not necessarily a subspace, since it need not be closed under addition.

Sums of Subspaces

Definition (Subspace Sum)

Let U_1, $U_2 \subseteq V$ be subspaces of a vector space V. Define the (**subspace**) **sum** of U_1 and U_2 to be the set

$$U_1 + U_2 = \{\mathbf{u}_1 + \mathbf{u}_2 \,|\, \mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2\}$$

Theorem

$U_1 + U_2$ is a subspace of V. In fact, $U_1 + U_2$ is the smallest subspace of V that contains both U_1 and U_2.

Subspace Sum: Example

Example

Let $U_1 = \{(x,0,0) \mid x \in \mathbb{R}\}$ and $U_2 = \{(0,y,0) \mid y \in \mathbb{R}\}$. Then

$$U_1 + U_2 = \{(x,y,0) \mid x,y \in \mathbb{R}\}.$$

Example

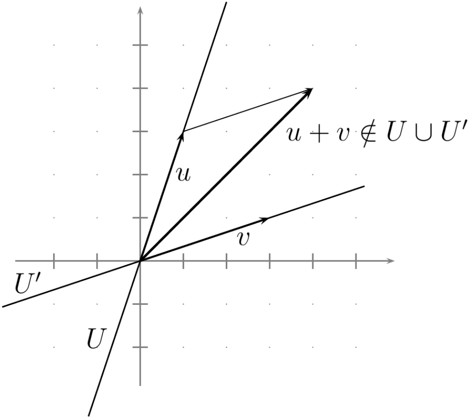
Let $U_1 = \{(x,0,0) \mid x \in \mathbb{R}\}$ and $U_2 = \{(y,y,0) \mid y \in \mathbb{R}\}$. Then

$$U_1 + U_2 = ?$$

Subspace Sum: Example

Example

$\mathbb{R}^2 = U + U'$.



The union $U \cup U'$ of two subspaces is not necessarily a subspace.

Direct Sums of Subspaces

Definition (Direct Sum of Subspaces)

Suppose every $\mathbf{u} \in U$ can be **uniquely** written as $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ for $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$. Then we use

$$U = U_1 \oplus U_2$$

to denote the **direct sum** of U_1 and U_2.

Direct Sum: Example

Example

Let $U_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ and $U_2 = \{(0, 0, z) \mid z \in \mathbb{R}\}$. Then

$$\mathbb{R}^3 = U_1 \oplus U_2.$$

Example

Let $U_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ and $U_2 = \{(0, w, z) \mid w, z \in \mathbb{R}\}$. Then

$$\mathbb{R}^3 = U_1 + U_2, \text{ but is not the direct sum of } U_1 \text{ and } U_2.$$

Verification of Direct Sum

Proposition 1

Let U_1, $U_2 \subseteq V$ be subspaces. Then $V = U_1 \oplus U_2$ if and only if the following two conditions hold:

1. $V = U_1 + U_2$;
2. If $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$ with $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$, then $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$.

Proposition 2

Let U_1, $U_2 \subseteq V$ be subspaces. Then $V = U_1 \oplus U_2$ if and only if the following two conditions hold:

1. $V = U_1 + U_2$;
2. $U_1 \cap U_2 = \{\mathbf{0}\}$;

Direct Sum: Example

Example

Let

$$U_1 = \{p \in P_{2n} \,|\, a_0 + a_2 t^2 + \cdots + a_{2n} t^{2n}\},$$
$$U_2 = \{p \in P_{2n} \,|\, a_1 t + a_3 t^3 + \cdots + a_{2n-1} t^{2n-1}\}.$$

Then

$$P_{2n} = U_1 \oplus U_2.$$

Symmetric Matrices

Example

- The **transpose** A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained by interchanging rows and columns of A, that is, $(A^t)_{ij} = A_{ji}$.
- A **symmetric matrix** A has $A^t = A$ and must be square.
- The set Sym of all symmetric matrices in M_n (i.e. $M_{n\times n}(F)$) is a subspace of M_n.

Example

- An $n \times n$ matrix A is a **diagonal matrix** if $A_{ij} = 0$ whenever $i \neq j$
- The set of diagonal matrices is a subspace of M_n.

Skew-Symmetric Matrices

Example

- A **skew-symmetric matrix** A has $A^t = -A$ and must be square.
- The set $\mathrm{SkewSym}$ of all skew-symmetric matrices in M_n is a subspace of M_n.

Direct Sum: Symmetric and Skew-Symmetric Matrices

Any matrix A can be written in the form

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) = B + C.$$

It is easy to verify that B is symmetric and C is skew-symmetric and so we have a decomposition of A as the sum of a symmetric matrix and a skew-symmetric matrix.

Since the sets Sym and $\mathrm{SkewSym}$ of all symmetric and skew-symmetric matrices in M_n are subspaces of M_n, we have

$$M_n = \mathrm{Sym} + \mathrm{SkewSym}$$

Since $\mathrm{Sym} \cap \mathrm{SkewSym} = \{\mathbf{0}\}$, we have

$$M_n = \mathrm{Sym} \oplus \mathrm{SkewSym}$$