Math 4377/6308 Advanced Linear Algebra

1.4 Linear Combinations & Systems of Linear Equations

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1.4 Linear Combinations & Systems of Linear Equations

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Linear Combinations

Definition

Let V be a vector space and S a nonempty subset of V. A vector $\mathbf{v} \in V$ is called a **linear combination** of vectors of S if there exist a finite number of vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\cdots$, $\mathbf{u}_n$ in S and scalars a_1, a_2, $\cdots$, a_n in F such that

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n.$$

In this case we also say that $\mathbf{v}$ is a **linear combination** of $\mathbf{u}_1$, $\mathbf{u}_2$, $\cdots$, $\mathbf{u}_n$ and call a_1, a_2, $\cdots$, a_n the **coefficients** of the linear combination

Note that $0\mathbf{v} = \mathbf{0}$ for each $\mathbf{v} \in V$, so the zero vector is a linear combination of any nonempty subset of V.

Linear Combinations of Vectors in $\mathbf{R}^2$

Parallelogram Rule for Addition of Two Vectors

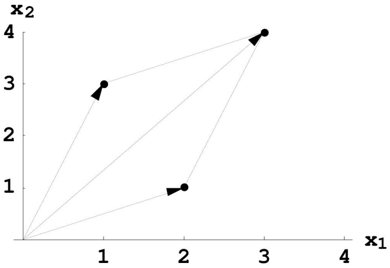
If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$.

Geometric Description of $\mathbf{R}^2$

Vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the point (x_1, x_2) in the plane. $\mathbf{R}^2$ is the set of all points in the plane.

Example

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
Graphs of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are:



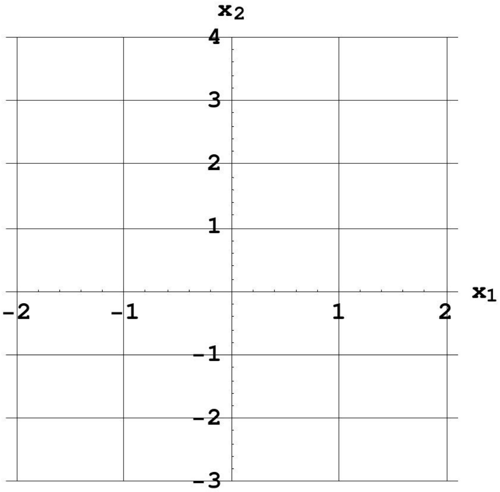
(Parallelogram Rule)

Linear Combinations of Vectors in $\mathbf{R}^2$ (cont.)

Example

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.

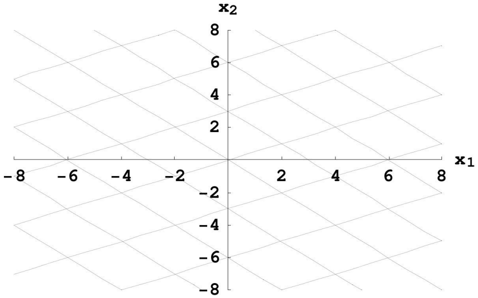


Linear Combinations of Vectors in $\mathbf{R}^2$: Example

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



Linear Combinations: Example

Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.
Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Solution: Vector $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if can we find weights x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Vector Equation (fill-in):

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

Linear Combinations: Example (cont.)

Corresponding System:

$$\begin{array}{ccccccc} x_1 & + & 4x_2 & + & 3x_3 & = & -1 \\ & & 2x_2 & + & 6x_3 & = & 8 \\ 3x_1 & + & 14x_2 & + & 10x_3 & = & -5 \end{array}$$

Corresponding Augmented Matrix:

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \implies \begin{array}{l} x_1 = \_\_\_ \\ x_2 = \_\_\_ \\ x_3 = \_\_\_ \end{array}$$

Linear Combinations: Review

Review of the last example: $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{matrix}$$

Solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

is found by solving the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}.$$

Linear Combinations and Vector Equation

Vector Equation

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix.

Solving a System of Linear Equations

Example

Solving a System in Matrix Form

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix}$$

(augmented matrix)

$$\downarrow$$

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \qquad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcr} x_1 & = & 3 \\ x_2 & = & 2 \end{array} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Operations

Elementary Row Operations

1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Row Equivalent Matrices

Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Solving a System by Row Eliminations: Example

Example (Row Eliminations to a Triangular Form)

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solving a System by Row Eliminations: Example (cont.)

Example (Row Eliminations to a Diagonal Form)

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rrrcr} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solution: $(29, 16, 3)$

Solving a System by Row Eliminations: Example (cont.)

Example (Check the Answer)

Is $(29, 16, 3)$ a solution of the **original** system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcll} (29) - 2(16) + (3) & = 29 - 32 + 3 & = & 0 \\ 2(16) - 8(3) & = 32 - 24 & = & 8 \\ -4(29) + 5(16) + 9(3) & = -116 + 80 + 27 & = & -9 \end{array}$$

Span of a Set of Vectors: Examples

Example

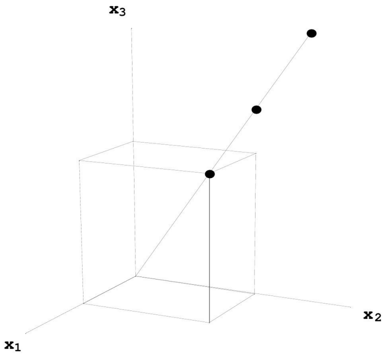
Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

Label the origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

together with

$\mathbf{v}$, $2\mathbf{v}$ and $1.5\mathbf{v}$

on the graph.



$\mathbf{v}$, $2\mathbf{v}$ and $1.5\mathbf{v}$ all lie on the same line.

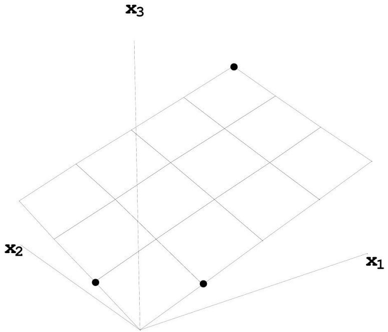
Span$\{\mathbf{v}\}$ is the set of all vectors of the form $c\mathbf{v}$.
Here, **Span**$\{\mathbf{v}\}$ = a line through the origin.

Span of a Set of Vectors: Examples (cont.)

Example

Label

$\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $3\mathbf{u}+4\mathbf{v}$ on the graph.



$\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $3\mathbf{u}+4\mathbf{v}$ all lie in the same plane.

Span$\{\mathbf{u},\mathbf{v}\}$ is the set of all vectors of the form $x_1\mathbf{u}+x_2\mathbf{v}$.
Here, **Span**$\{\mathbf{u},\mathbf{v}\}$ = a plane through the origin.

Span of a Set of Vectors: Definition

Span of a Set of Vectors

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbf{R}^n$; then

$$\textbf{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = \text{set of all linear combinations of } \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p.$$

Span of a Set of Vectors (Stated another way)

$\textbf{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written as

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p$$

where $x_1, x_2, \ldots, x_p$ are scalars.

Span of a Set of Vectors in $\mathbb{R}^2$

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(a) Find a vector in $\textbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) Describe $\textbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ geometrically.

Spanning Sets in $\mathbf{R}^3$

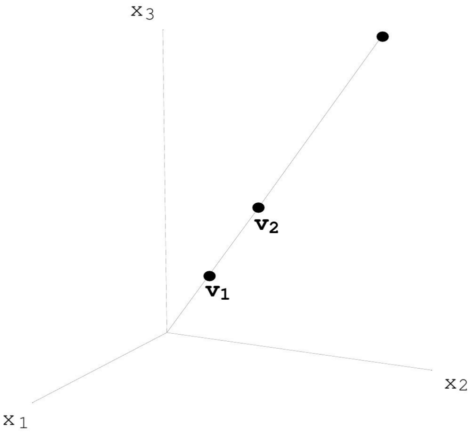
Example

$\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$

$$\mathbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbf{Span}\{\mathbf{v}_1\}$$

$$= \mathbf{Span}\{\mathbf{v}_2\}$$

(line through the origin)

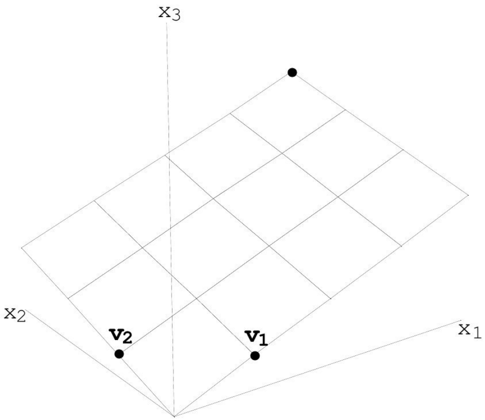


Spanning Sets in $\mathbf{R}^3$ (cont.)

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$.

Is $\mathbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ a line or a plane?



$\mathbf{v}_2$ is **not** a multiple of $\mathbf{v}_1$

$\mathbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ =plane through the origin

Spanning Sets

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is $\mathbf{b}$ in the plane spanned by the columns of A?

Solution: ? Do x_1 and x_2 exist so that

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Corresponding augmented matrix:

$$\begin{bmatrix} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{bmatrix}$$

So $\mathbf{b}$ is not in the plane spanned by the columns of A

A Shortcut for Determining Subspaces

Theorem (1)

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof: In order to verify this, check properties a, b and c of definition of a subspace.

a. $\mathbf{0}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ since

$$\mathbf{0} = \_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\mathbf{v}_p$$

b. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under vector addition, we choose two arbitrary vectors in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

A Shortcut for Determining Subspaces (cont.)

Then

$$\mathbf{u}+\mathbf{v} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= (\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_1) + (\_\_\_\_\mathbf{v}_2 + \_\_\_\_\mathbf{v}_2) + \cdots + (\_\_\_\_\mathbf{v}_p + \_\_\_\_\mathbf{v}_p)$$

$$= (a_1 + b_1)\,\mathbf{v}_1 + (a_2 + b_2)\,\mathbf{v}_2 + \cdots + (a_p + b_p)\,\mathbf{v}_p.$$

So $\mathbf{u}+\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

c. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

A Shortcut for Determining Subspaces (cont.)

Then

$$c\mathbf{v} = c\left(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p\right)$$

$$= \_\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\_\mathbf{v}_p$$

So $c\mathbf{v}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Since properties a, b and c hold, $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Determining Subspaces: Recap

Recap

1. To show that H is a subspace of a vector space, use Theorem 1.

2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms a, b or c (from the definition of a subspace) is violated.

Determining Subspaces: Example

Example

Is $V = \{(a+2b, 2a-3b) : \ a \text{ and } b \text{ are real}\}$ a subspace of $\mathbf{R}^2$? Why or why not?

Solution: Write vectors in V in column form:

$$\begin{bmatrix} a+2b \\ 2a-3b \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} 2b \\ -3b \end{bmatrix}$$

$$= \_\_\_\_\_ \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \_\_\_\_\_ \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

So $V =$Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ and therefore V is a subspace of _____ by Theorem 1.

Determining Subspaces: Example

Example

Is $H = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} : \ a \text{ and } b \text{ are real} \right\}$ a subspace of $\mathbf{R}^3$?
Why or why not?

Solution: $\mathbf{0}$ is not in H since $a = b = 0$ or any other combination of values for a and b does not produce the zero vector. So property _____ fails to hold and therefore H is not a subspace of $\mathbf{R}^3$.

Determining Subspaces: Example

Example

Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2\times 2}$? Explain.

Solution: Since

$$\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$$

$$= a\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} + b\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

Therefore $H =$Span $\left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}$ and so H is a subspace of $M_{2\times 2}$.

Spanning Sets

Theorem (1.5)

The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V that contains S must also contain the span of S.

Definition

The **subspace spanned** (or **subspace generated**) by a nonempty set S of vectors in V is the set of all linear combinations of vectors from S:

$$< S >= \mathbf{span}(S) = \{c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \,|\, c_i \in F, \mathbf{v}_i \in S\}$$

When $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a finite set, we use the notation $< \mathbf{v}_1, \cdots, \mathbf{v}_n >$ or $\mathbf{span}(\mathbf{v}_1, \cdots, \mathbf{v}_n)$. A set S of vectors in V is said to **span** V , or **generate** V, if $V = \mathbf{span}(S)$.