Math 4377/6308 Advanced Linear Algebra

2.2 Properties of Linear Transformations, Matrices.

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2.2 Properties of Linear Transformations, Matrices.

- Null Spaces and Ranges
- Injective, Surjective, and Bijective
- Dimension Theorem
- Nullity and Rank
- Linear Map and Values on Basis
- Coordinate Vectors
- Matrix Representations

Theorem (2.1-a)

Let $T : V \to W$ be a linear map. Then $\mathrm{null}(T)$ is a subspace of V.

Proof. We need to show that $0 \in \mathrm{null}\,(T)$ and that $\mathrm{null}\,(T)$ is closed under addition and scalar multiplication. By linearity, we have

$$T(0) = T(0+0) = T(0) + T(0)$$

so that $T(0) = 0$. Hence $0 \in \mathrm{null}\,(T)$. For closure under addition, let $u, v \in \mathrm{null}\,(T)$. Then

$$T(u+v) = T(u) + T(v) = 0 + 0 = 0,$$

and hence $u + v \in \mathrm{null}\,(T)$. Similarly, for closure under scalar multiplication, let $u \in \mathrm{null}\,(T)$ and $a \in \mathbb{F}$. Then

$$T(au) = aT(u) = a0 = 0,$$

and so $au \in \mathrm{null}\,(T)$. □

Injective, Surjective, and Bijective Linear Maps

Definition

The linear map $T : V \to W$ is called **injective** (**one-to-one**) if, for all $u, v \in V$, the condition $Tu = Tv$ implies that $u = v$. In other words, different vectors in V are mapped to different vectors in W.

Definition

The linear map $T : V \to W$ is called **surjective** (**onto**) if $\mathrm{range}(T) = W$.

Definition

A linear map $T : V \to W$ is called **bijective** if T is both injective and surjective.

Injective Linear Map

Theorem (2.4)

Let $T : V \to W$ be a linear map. Then T is injective if and only if $\mathrm{null}(T) = \{0\}$.

Proof.
("$\Longrightarrow$") Suppose that T is injective. Since $\mathrm{null}\,(T)$ is a subspace of V, we know that $0 \in \mathrm{null}\,(T)$. Assume that there is another vector $v \in V$ that is in the kernel. Then $T(v) = 0 = T(0)$. Since T is injective, this implies that $v = 0$, proving that $\mathrm{null}\,(T) = \{0\}$.
("$\Longleftarrow$") Assume that $\mathrm{null}\,(T) = \{0\}$, and let $u, v \in V$ be such that $Tu = Tv$. Then $0 = Tu - Tv = T(u - v)$ so that $u - v \in \mathrm{null}\,(T)$. Hence $u - v = 0$, or, equivalently, $u = v$. This shows that T is indeed injective. □

Linear Maps and Ranges

Theorem (2.1-b)

Let $T : V \to W$ be a linear map. Then $\operatorname{range}(T)$ is a subspace of V.

Proof. We need to show that $0 \in \operatorname{range}(T)$ and that $\operatorname{range}(T)$ is closed under addition and scalar multiplication. We already showed that $T0 = 0$ so that $0 \in \operatorname{range}(T)$.

For closure under addition, let $w_1, w_2 \in \operatorname{range}(T)$. Then there exist $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$. Hence

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2,$$

and so $w_1 + w_2 \in \operatorname{range}(T)$.

For closure under scalar multiplication, let $w \in \operatorname{range}(T)$ and $a \in \mathbb{F}$. Then there exists a $v \in V$ such that $Tv = w$. Thus

$$T(av) = aTv = aw,$$

and so $aw \in \operatorname{range}(T)$. □

Dimension Theorem

Theorem (2.3, Dimension Theorem)

Let V be a finite-dimensional vector space and $T : V \to W$ be a linear map. Then $\mathrm{range}(T)$ is a finite-dimensional subspace of W and

$$\dim(V) = \dim(\mathrm{null}(T)) + \dim(\mathrm{range}(T)).$$

Proof. Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Since null (T) is a subspace of V, we know that null (T) has a basis $(u_1, \ldots, u_m)$. This implies that $\dim(\text{null}\,(T)) = m$. By the Basis Extension Theorem, it follows that $(u_1, \ldots, u_m)$ can be extended to a basis of V, say $(u_1, \ldots, u_m, v_1, \ldots, v_n)$, so that $\dim(V) = m + n$.

The theorem will follow by showing that $(Tv_1, \ldots, Tv_n)$ is a basis of range (T) since this would imply that range (T) is finite-dimensional and $\dim(\text{range}\,(T)) = n$, proving Equation (6.4).

Dimension Theorem (cont.)

Since $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ spans V, every $v \in V$ can be written as a linear combination of these vectors; i.e.,

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n,$$

where $a_i, b_j \in \mathbb{F}$. Applying T to v, we obtain

$$Tv = b_1Tv_1 + \cdots + b_nTv_n,$$

where the terms Tu_i disappeared since $u_i \in \operatorname{null}(T)$. This shows that $(Tv_1, \ldots, Tv_n)$ indeed spans $\operatorname{range}(T)$.

To show that $(Tv_1, \ldots, Tv_n)$ is a basis of $\operatorname{range}(T)$, it remains to show that this list is linearly independent. Assume that $c_1, \ldots, c_n \in \mathbb{F}$ are such that

$$c_1Tv_1 + \cdots + c_nTv_n = 0.$$

Dimension Theorem (cont.)

By linearity of T, this implies that

$$T(c_1 v_1 + \cdots + c_n v_n) = 0,$$

and so $c_1 v_1 + \cdots + c_n v_n \in \operatorname{null}(T)$. Since $(u_1, \ldots, u_m)$ is a basis of $\operatorname{null}(T)$, there must exist scalars $d_1, \ldots, d_m \in \mathbb{F}$ such that

$$c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m.$$

However, by the linear independence of $(u_1, \ldots, u_m, v_1, \ldots, v_n)$, this implies that all coefficients $c_1 = \cdots = c_n = d_1 = \cdots = d_m = 0$. Thus, $(Tv_1, \ldots, Tv_n)$ is linearly independent, and we are done. $\square$

Surjective Linear Map

Corollary

Let $T : V \to W$ be a linear map.

1. If $\dim(V) > \dim(W)$, then T is not injective.
2. If $\dim(V) < \dim(W)$, then T is not surjective.

Proof. By Theorem 6.5.1, we have that

$$\begin{aligned}\dim(\operatorname{null}(T)) &= \dim(V) - \dim(\operatorname{range}(T)) \\ &\geq \dim(V) - \dim(W) > 0.\end{aligned}$$

Since T is injective if and only if $\dim(\operatorname{null}(T)) = 0$, T cannot be injective.

Similarly,

$$\begin{aligned}\dim(\operatorname{range}(T)) &= \dim(V) - \dim(\operatorname{null}(T)) \\ &\leq \dim(V) < \dim(W),\end{aligned}$$

and so $\operatorname{range}(T)$ cannot be equal to W. Hence, T cannot be surjective. $\square$

Nullity and Rank

Definition

For vector spaces V, W and linear $T : V \to W$, if $\mathrm{null}(T)$, i.e., $N(T)$, and $\mathrm{range}(T)$, i.e, $R(T)$, are finite-dimensional, the nullity and the rank of T are the dimensions of $\mathrm{null}(T)$ and $\mathrm{range}(T)$, respectively.

Theorem (Dimension Theorem, 2.3)

For vector spaces V, W and linear $T : V \to W$, if V is finite-dimensional then

$$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V)$$

Rank of a Matrix

Rank

The **rank** of A is the dimension of the column space of A.

$$\boxed{\text{rank } A = \dim \text{Col } A = \# \text{ of pivot columns of } A = \dim \text{Row } A}.$$

The set of all linear combinations of the row vectors of a matrix A is called the **row space** of A and is denoted by Row A.

$$\boxed{\text{Col } A^T = \text{Row } A}.$$

Note the following:

- $\dim \text{Col } A = \#$ of pivots of $A = \dim \text{Row } A$.
- $\dim \text{Nul } A = \#$ of free variables $= \#$ of nonpivot columns of A.

Rank Theorem

$$\underbrace{\text{rank } A}_{\displaystyle\updownarrow} + \underbrace{\dim \text{Nul } A}_{\displaystyle\updownarrow} = \underbrace{n}_{\displaystyle\updownarrow}$$

$$\left\{\begin{array}{c}\#\text{ of pivot}\\ \text{columns}\\ \text{of } A\end{array}\right\} \quad \left\{\begin{array}{c}\#\text{ of nonpivot}\\ \text{columns}\\ \text{of } A\end{array}\right\} \quad \left\{\begin{array}{c}\#\text{ of}\\ \text{columns}\\ \text{of } A\end{array}\right\}$$

Theorem (Rank Theorem)

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{ Nul } A = n.$$

Rank Theorem: Example

Since Row $A =$ Col A^T, $\boxed{\text{rank } A = \text{rank } A^T}$.

Example

Suppose that a 5×8 matrix A has rank 5. Find $\dim$ Nul A, $\dim$ Row A and rank A^T. Is Col $A = \mathbf{R}^5$?

Solution:

$$\underbrace{\text{rank } A}_{\updownarrow \atop 5} + \underbrace{\dim \text{Nul } A}_{\downarrow \atop ?} = \underbrace{n}_{\updownarrow \atop 8}$$

$$5 + \dim \text{Nul } A = 8 \quad \Rightarrow \quad \dim \text{Nul } A = \_\_\_\_\_$$

$$\dim \text{Row } A = \text{rank } A = \_\_\_\_\_$$

$$\Rightarrow \quad \text{rank } A^T = \text{rank } \_\_\_\_\_ = \_\_\_\_\_$$

Since rank $A = \#$ of pivots in $A = 5$, there is a pivot in every row. So the columns of A span $\mathbf{R}^5$. Hence Col $A = \mathbf{R}^5$.

Rank Theorem: Example

Example

For a 9×12 matrix A, find the smallest possible value of $\dim$ Nul A.

Solution:

$$\text{rank } A + \dim \text{Nul } A = 12$$

$$\dim \text{Nul } A = 12 - \underbrace{\text{rank } A}_{\text{largest possible value=\_\_\_\_\_}}$$

smallest possible value of $\dim$ Nul A = _____

Theorem (2.5)

For vector spaces V, W of equal (finite) dimension and linear $T : V \to W$, the following are equivalent:

(a) *T is one-to-one.*

(b) *T is onto.*

(c) $\operatorname{rank}(T) = \dim(V)$

Linear Map and Values on Basis

Theorem (2.6)

Let $(v_1, \cdots, v_n)$ be a basis of V and $(w_1, \cdots, w_n)$ be an arbitrary list of vectors in W. Then there exists a unique linear map $T : V \to W$ such that $T(v_i) = w_i$, $\forall i = 1, 2, \cdots, n$.

Proof. First we verify that there is at most one linear map T with $T(v_i) = w_i$. Take any $v \in V$. Since $(v_1, \ldots, v_n)$ is a basis of V there are unique scalars $a_1, \ldots, a_n \in \mathbb{F}$ such that $v = a_1 v_1 + \cdots + a_n v_n$. By linearity, we have

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n) = a_1 w_1 + \cdots + a_n w_n, \qquad (6.3)$$

and hence $T(v)$ is completely determined. To show existence, use Equation (6.3) to define T. It remains to show that this T is linear and that $T(v_i) = w_i$. These two conditions are not hard to show and are left to the reader. □

Corollary

Suppose $\{v_1, \cdots, v_n\}$ is a finite basis for V, then if U, $T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, \cdots, n$, then $U = T$.

Example

Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Suppose $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation which maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: First, note that

$$T(\mathbf{e}_1) = \_\_\_\_\_\_ \qquad \text{and} \qquad T(\mathbf{e}_2) = \_\_\_\_\_\_.$$

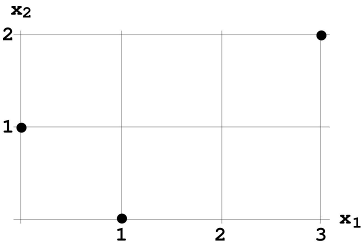
Also

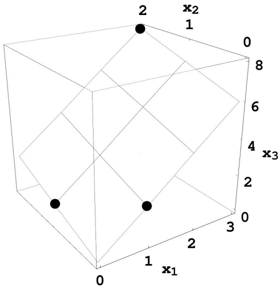
$$\_\_\_\mathbf{e}_1 + \_\_\_\mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Then

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T\left(\_\_\_\mathbf{e}_1 + \_\_\_\mathbf{e}_2\right) =$$

$$\_\_\_T\left(\mathbf{e}_1\right) + \_\_\_T\left(\mathbf{e}_2\right) =$$





$$T\left(3\mathbf{e}_1 + 2\mathbf{e}_2\right) = 3T\left(\mathbf{e}_1\right) + 2T\left(\mathbf{e}_2\right)$$

Matrix Transformations: Example (cont.)

Also

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\_\_\_\_\_\mathbf{e}_1 + \_\_\_\_\_\mathbf{e}_2\right) =$$

$$\_\_\_\_T\left(\mathbf{e}_1\right) + \_\_\_\_T\left(\mathbf{e}_2\right) =$$

Coordinate Vectors

Definition

For a finite-dimensional vector space V, an **ordered basis** for V is a basis for V with a specific order. In other words, it is a finite sequence of linearly independent vectors in V that generates V.

Definition

Let $\beta = \{u_1, \cdots, u_n\}$ be an ordered basis for V, and for $x \in V$ let $a_1, \cdots, a_n$ be the unique scalars such that

$$x = \sum_{i=1}^{n} a_i u_i.$$

The **coordinate vector** of x relative to β is

$$[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Example

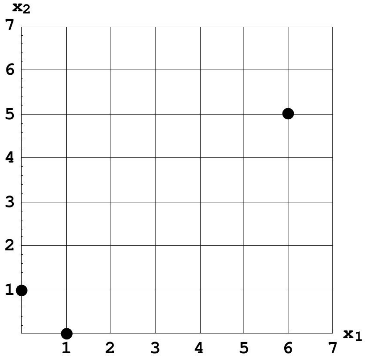
Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and let $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution:

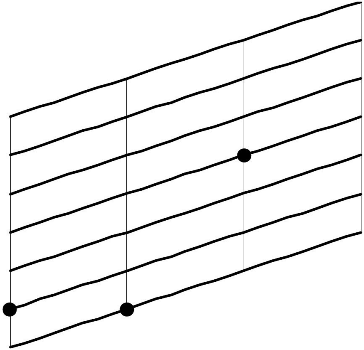
If $[\mathbf{x}]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\mathbf{x} =$ ____ $\begin{bmatrix} 3 \\ 1 \end{bmatrix} +$ ____ $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$.

If $[\mathbf{x}]_E = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$, then $\mathbf{x} =$ ____ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} +$ ____ $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$.

Coordinate Systems: Example (cont.)



Standard graph paper



$\beta-$ graph paper

Matrix Representations

Definition

Suppose V, W are finite-dimensional vector spaces with ordered bases $\beta = \{v_1, \cdots, v_n\}$, $\gamma = \{w_1, \cdots, w_m\}$. For linear $T : V \to W$, there are unique scalars $a_{ij} \in F$ such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad \text{for } 1 \leq j \leq n.$$

The $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ is the **matrix representation** of T in the ordered bases β and γ, written $A = [T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$, then $A = [T]_\beta$.

Note that the jth column of A is $[T(v_j)]_\gamma$, and if $[U]_\beta^\gamma = [T]_\beta^\gamma$ for linear $U : V \to W$, then $U = T$.

Example

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$$

Solution:

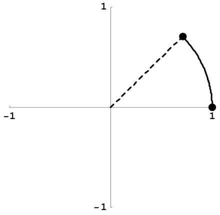
$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \text{standard matrix of the linear transformation } T$$

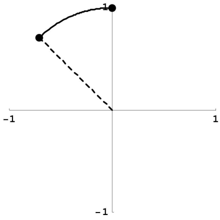
$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \qquad \text{(fill-in)}$$

Matrix of Linear Transformation: Example

Example

Find the standard matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates a point about the origin through an angle of $\frac{\pi}{4}$ radians (counterclockwise).





$$\downarrow$$

$$A = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$T(\mathbf{e}_1) = \begin{bmatrix} \\ \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} \\ \end{bmatrix}$$

Identity Matrix

Identity Matrix

I_n is an $n \times n$ matrix with 1's on the main left to right diagonal and 0's elsewhere. The ith column of I_n is labeled $\mathbf{e}_i$.

Example

$$I_3 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$I_3\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \_\_\_\_ \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \_\_\_\_ \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \_\_\_\_ \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \_\_\_\_.$$

Addition and Scalar Multiplication

Definition

Let T, $U : V \to W$ be arbitrary functions of vector spaces V, W over F. Then $T + U$, $aT : V \to W$ are defined by $(T + U)(x) = T(x) + U(x)$ and $(aT)(x) = aT(x)$, respectively, for all $x \in V$ and $a \in F$.

Theorem (2.7)

With the operations defined above, for vector spaces V, W over F and linear T, $U : V \to W$:

(a) *$aT + U$ is linear for all $a \in F$*

(b) *The collection of all linear transformations from V to W is a vector space over F*

Definition

For vector spaces V, W over F, the vector space of all linear transformations from V into W is denoted by $\mathcal{L}(V, W)$, or just $\mathcal{L}(V)$ if $V = W$.

Theorem (2.8)

For finite-dimensional vector spaces V, W with ordered bases β, γ, and linear transformations T, $U : V \to W$:

(a) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.

(b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ *for all scalars* a.