Math 4377/6308 Advanced Linear Algebra

3.3 Systems of Linear Equations – Theoretical Aspects

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3.3 Systems of Linear Equations – Theoretical Aspects

- Systems of Linear Equations
- Solution Sets: Homogeneous System
- Solution Sets: Nonhomogeneous System
- Invertibility
- Consistency

Systems of Linear Equations

System of m linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

or

$$Ax = b$$

with coefficient matrix A and vectors x, b:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Solution Sets

- A solution to the system $Ax = b$:

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in F^n \quad \text{such that } As = b.$$

- The solution set of the system: The set of all solutions
- Consistent system: Nonempty solution set
- Inconsistent system: Empty solution set

Solution Sets: Homogeneous System

Definition

$Ax = b$ is homogeneous if $b = 0$, otherwise nonhomogeneous.

Theorem (3.8)

Let $Ax = 0$ be a homogeneous system of m equations in n unknowns. The set of all solutions to $Ax = 0$ is $K = N(L_A)$, which is a subspace of F^n of dimension $n - \mathrm{rank}(L_A) = n - \mathrm{rank}(A)$.

Homogeneous System: Trivial Solutions

Example

$$\begin{array}{rcrcl} x_1 & + & 10x_2 & = & 0 \\ 2x_1 & + & 20x_2 & = & 0 \end{array}$$

Corresponding matrix equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Trivial solution: $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\mathbf{x} = \mathbf{0}$

Homogeneous System: Nontrivial Solutions

The homogeneous system $A\mathbf{x} = \mathbf{0}$ *always* has the **trivial solution**, $\mathbf{x} = \mathbf{0}$.

Nontrivial Solution

Nonzero vector solutions are called **nontrivial solutions.**

Example (cont.)

Do **nontrivial** solutions exist?

$$\begin{bmatrix} 1 & 10 & 0 \\ 2 & 20 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consistent system with a free variable has infinitely many solutions.

A homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions if and only if the system of equations has

____________________________________.

Homogeneous System: Example 1

Example (1)

Determine if the following homogeneous system has nontrivial solutions and then describe the solution set.

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & - & 6x_3 & = & 0 \\ 4x_1 & + & 8x_2 & - & 10x_3 & = & 0 \end{array}$$

Solution: There is at least one free variable (why?)

$\Longrightarrow$ nontrivial solutions exist

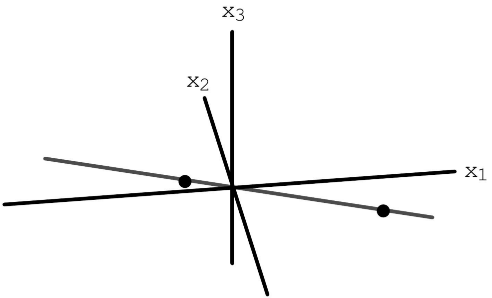
$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Longrightarrow \begin{array}{l} x_1 = \\ x_2 \quad \text{is free} \\ x_3 = \end{array}$$

Homogeneous System: Example 1 (cont.)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{v}$$

Graphical representation:



solution set $=$ span$\{\mathbf{v}\}$ $=$ line through $\mathbf{0}$ in $\mathbf{R}^3$

Homogeneous System: Non Trivial Solutions

Corollary

If $m < n$, the system $Ax = 0$ has a nonzero solution.

Solution Sets: Nonhomogeneous System

Theorem (3.9)

Let K be the solution set of $Ax = b$, and let K_H be the solution set of the corresponding homogeneous system $Ax = 0$. Then for any solution s to $Ax = b$:

$$K = \{s\} + K_H = \{s + k : k \in K_H\}.$$

Nonhomogeneous System: Example 2

Example (2)

Describe the solution set of

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 6x_3 & = & 0 \\ 4x_1 & + & 8x_2 & - & 10x_3 & = & 4 \end{array}$$

(same left side as in the previous example)

Solution:

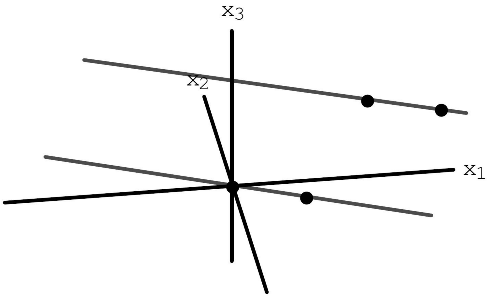
$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

Nonhomogeneous System: Example 2 (cont.)

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2\mathbf{v}$$

Graphical representation:



Parallel solution sets of $A\mathbf{x} = \mathbf{0}$ & $A\mathbf{x} = \mathbf{b}$

Nonhomogeneous System: Recap of Previous Two Examples

Example (1. Solution of $A\mathbf{x} = \mathbf{0}$)

$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{v}$$

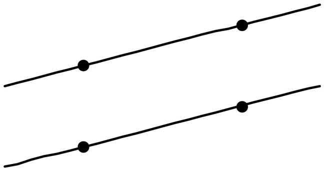
$\mathbf{x} = x_2\mathbf{v} =$ parametric equation of line passing through $\mathbf{0}$ and $\mathbf{v}$

Example (2. Solution of $A\mathbf{x} = \mathbf{b}$)

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2\mathbf{v}$$

$\mathbf{x} = \mathbf{p} + x_2\mathbf{v} =$ parametric equation of line passing through $\mathbf{p}$ parallel to $\mathbf{v}$

Nonhomogeneous System



Parallel solution sets of
$A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Nonhomogeneous System: Example

Example

Describe the solution set of $2x_1 - 4x_2 - 4x_3 = 0$; compare it to the solution set $2x_1 - 4x_2 - 4x_3 = 6$.

Solution: Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 0$:

$$\begin{bmatrix} 2 & -4 & -4 & 0 \end{bmatrix} \sim \qquad \text{(fill-in)}$$

Vector form of the solution:

$$\mathbf{v} = \begin{bmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

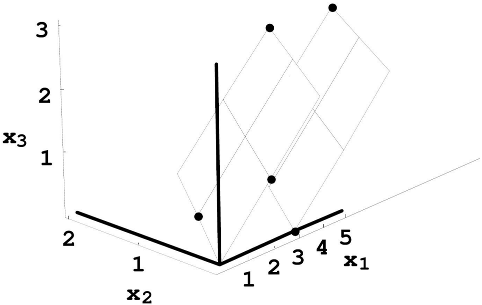
Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 6$:

$$\begin{bmatrix} 2 & -4 & -4 & 6 \end{bmatrix} \sim \qquad \text{(fill -in)}$$

Nonhomogeneous System: Example (cont.)

Vector form of the solution:

$$\mathbf{v} = \begin{bmatrix} 3 + 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$



Parallel Solution Sets of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

Invertibility

Theorem (3.10)

If A is invertible then the system $Ax = b$ has exactly one solution $x = A^{-1}b$. Conversely, if the system has exactly one solution then A is invertible.

Consistency

Theorem (3.11)

The system $Ax = b$ is consistent if and only if

$$\mathrm{rank}(A) = \mathrm{rank}(A|b)$$