Math 4377/6308 Advanced Linear Algebra

3.4 Systems of Linear Equations – Computational Aspects

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3.4 Systems of Linear Equations – Computational Aspects

- Equivalent Systems
- Reduced Row Echelon Form
- Gaussian Elimination
- General Solutions
- Interpretation of the Reduced Row Echelon Form

Equivalent Systems

Definition

Two systems of linear equations are called **equivalent** if they have the same solution set.

Theorem (3.13)

For $m \times n$ linear system $Ax = b$ and invertible $m \times m$ matrix C, the system $(CA)x = Cb$ is equivalent to $Ax = b$.

Equivalent Systems

Corollary

For linear system $Ax = b$, if $(A'|b')$ is obtained from $(A|b)$ by a finite number of elementary row operations, then $A'x = b'$ is equivalent to the original system.

Reduced Row Echelon Form

Definition

A matrix is in reduced row echelon form if:

(a) Any row containing a nonzero entry precedes any row in which all the entries are zero

(b) The first nonzero entry in each row is the only nonzero entry in its column

(c) The first nonzero entry in each row is 1 and it occurs in a column right of the first nonzero entry in the preceding row.

Example

$$\begin{pmatrix} 1 & 0 & x & 0 & x & 0 & x & x \\ & 1 & x & 0 & x & 0 & x & x \\ & & & 1 & x & 0 & x & x \\ & & & & & 1 & x & x \\ & & & & & & & \\ & & & & & & & \end{pmatrix}$$

Gaussian Elimination

Definition (Gaussian Elimination)

Reducing an augmented matrix to reduced row echelon form:

- In the **forward pass**, the matrix is transformed into upper triangular form where first nonzero entry of each row is 1, in a column to the right of the first nonzero entry of preceding rows.
- In the **backward pass** or **back-substitution**, the matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.

Pivots

Important Terms

- **pivot position:** a position of a leading entry in an echelon form of the matrix.
- **pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's.
- **pivot column:** a column that contains a pivot position.

Reduced Echelon Form: Examples

Example (Row reduce to echelon form and locate the pivots)

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution

pivot

$\swarrow$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$\uparrow$

pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Possible Pivots:

Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form (cont.))

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Original Matrix: } \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$\begin{array}{lccc} & \uparrow & \uparrow & \uparrow \\ \text{pivot columns:} & 1 & 2 & 4 \end{array}$$

Note

There is no more than one pivot in any row. There is no more than one pivot in any column.

Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form and then to REF)

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form and then to REF (cont.))

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \text{ (echelon form)}$$

Reduced Echelon Form: Examples (cont.)

Example (Row reduce to echelon form and then to REF (cont.))

Final step to create the reduced echelon form:

Beginning with the rightmost leading entry, and working upwards to the left, create zeros above each leading entry and scale rows to transform each leading entry into 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Gaussian Elimination

Theorem (3.14)

Gaussian elimination transforms any matrix into its reduced row echelon form.

Solutions of Linear Systems

Important Terms

- **basic variable:** any variable that corresponds to a pivot column in the augmented matrix of a system.
- **free variable:** all nonbasic variables.

Example (Solutions of Linear Systems)

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad +6x_2 \qquad\qquad +3x_4 \qquad &= 0 \\ x_3 \quad -8x_4 \qquad &= 5 \\ x_5 \quad &= 7 \end{aligned}$$

pivot columns:
basic variables:
free variables:

Solutions of Linear Systems (cont.)

Final Step in Solving a Consistent Linear System

After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations, *Solve each equation for the basic variable in terms of the free variables (if any) in the equation.*

Example (General Solutions of Linear Systems)

$$\begin{array}{cccccl} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array}$$

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

(general solution)

Warning

Use only the reduced echelon form to solve a system.

General Solutions of Linear Systems

General Solution

The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.)

Example (General Solutions of Linear Systems (cont.))

$$x_1 = -6x_2 - 3x_4$$
$$x_2 \text{ is free}$$
$$x_3 = 5 + 8x_4$$
$$x_4 \text{ is free}$$
$$x_5 = 7$$

The above system has **infinitely many solutions.** Why?

General Solutions

Theorem (3.15)

Let $Ax = b$ be a system of r nonzero equations in n unknowns. Suppose $rank(A) = \mathrm{rank}(A|b)$ and that $(A|b)$ is in reduced row echelon form. Then

(a) $rank(A) = r$.

(b) *If the general solution is of the form*

$$s = s_0 + t_1 u_1 + t_2 u_2 + \cdots + t_{n-r} u_{n-r}$$

then $\{u_1, u_2, \cdots, u_{n-r}\}$ is a basis for the solution set of the corresponding homogeneous system, and s_0 is a solution to the original system.

Interpretation of the Reduced Row Echelon Form

Theorem (3.16)

Let A be an $m \times n$ matrix of rank $r > 0$ and B the reduced row echelon form of A. Then

(a) *The number of nonzero rows in B is r.*

(b) *For each $i = 1, \cdots, r$, there is a column b_{j_i} of B s.t. $b_{j_i} = e_i$*

(c) *The columns of A numbered $j_1, \cdots, j_r$ are linearly independent*

(d) *For each $k = 1, \cdots, n$, if column k of B is $d_1 e_1 + \cdots + d_r e_r$ then column k of A is $d_1 a_{j_1} + \cdots + d_r a_{j_r}$*

Interpretation of the Reduced Row Echelon Form

Corollary

The reduced row echelon form of a matrix is unique.