Math 4377/6308 Advanced Linear Algebra

5.1 Eigenvalues and Eigenvectors

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5.1 Eigenvalues and Eigenvectors

- Diagonalization
- Eigenvalues and Eigenvectors
- Characteristic Polynomial
- Properties

Diagonalization

Definition

A linear operator T on a finite-dimensional vector space V is diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is diagonalizable if L_A is diagonalizable.

Eigenvalues and Eigenvectors

Definition

Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is an eigenvector of T if there exists a scalar eigenvalue λ corresponding to the eigenvector v such that $T(v) = \lambda v$.

Let $A \in M_{n \times n}(F)$. A nonzero vector $v \in F^n$ is an eigenvector of A if v is an eigenvector of L_A; that is, if $Av = \lambda v$ for some scalar eigenvalue λ of A corresponding to the eigenvector v.

Eigenvalues and Eigenvectors: Example

Example

Let $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Examine the images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A.

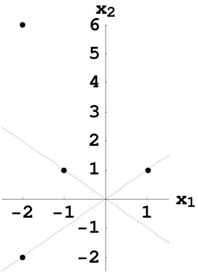
Solution

$$A\mathbf{u} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2\mathbf{u}$$

$\mathbf{u}$ is called an *eigenvector* of A since $A\mathbf{u}$ is a multiple of $\mathbf{u}$.

$$A\mathbf{v} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \neq \lambda\mathbf{v}$$

$\mathbf{v}$ is not an eigenvector of A since $A\mathbf{v}$ is not a multiple of $\mathbf{v}$.



$A\mathbf{u} = -2\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$

Eigenvalues and Eigenvectors: Example

Example

Show that 4 is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

Solution: Scalar 4 is an eigenvalue of A if and only if $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution.

$$A\mathbf{x} - 4\mathbf{x} = \mathbf{0}$$

$$A\mathbf{x} - 4\left(\_\_\_\_\right)\mathbf{x} = \mathbf{0}$$

$$\left(A - 4I\right)\mathbf{x} = \mathbf{0}.$$

To solve $\left(A - 4I\right)\mathbf{x} = \mathbf{0}$, we need to find $A - 4I$ first:

$$A - 4I = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}$$

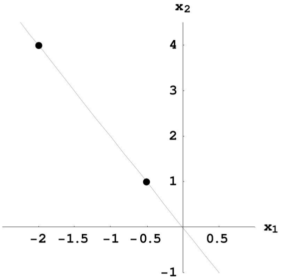
Eigenvalues and Eigenvectors: Example

Now solve $(A-4I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -4 & -2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \qquad \mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Each vector of the form $x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 4$.



Eigenspace for $\lambda = 4$

The set of all solutions to $(A-\lambda I)\mathbf{x} = \mathbf{0}$ is called the **eigenspace** of A corresponding to λ.

Diagonalization

Theorem (5.1)

A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T. If T is diagonalizable, $\beta = \{v_1, \cdots, v_n\}$ is an ordered basis of eigenvectors of T, and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leq j \leq n$.

Diagonalization

To diagonalize a matrix or a linear operator is to find a basis of eigenvectors and the corresponding eigenvalues.

Characteristic Polynomial

Theorem (5.2)

Let $A \in M_{n\times n}(F)$*. Then a scalar* λ *is an eigenvalue of* A *if and only if* $\det(A - \lambda I_n) = 0$.

Characteristic Polynomial

Definition

Let $A \in M_{n\times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the characteristic polynomial of A.

Characteristic Polynomial

Definition

Let T be a linear operator on an n-dimensional vector space V with ordered basis β. We define the characteristic polynomial $f(t)$ of T to be the characteristic polynomial of $A = [T]_\beta$:
$f(t) = \det(A - tI_n)$.

Properties

Theorem (5.3)

Let $A \in M_{n\times n}(F)$.

(a) *The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.*

(b) *A has at most n distinct eigenvalues.*

Properties

Theorem (5.4)

Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.