Numerical Analysis II

Exam 2 **MATH 6371-13338 (S2013)** **April 25 – April 28, 2013**

Name and ID: ____

- This is a 72 hour take-home exam. Please turn it in to me (Jiwen He) in a digital form at jiwenhe@math.uh.edu by 3:00pm on Sunday, April 28, i.e., 72 hours after you receive it. The exam is not supposed to take anything close to 72 hours; I am giving you 72 hours so you can take a number of long breaks (to eat, sleep, study for other classes, takes other exams, etc.).
- You may use any books, notes, but you may not discuss the exam with anyone until April 28, after everyone has taken the exam.
- Please make a copy of your exam before handing it in.
- The problems are all equally weighted. This is to make things simple, and not because the problems are all of equal difficulty.
- Keep in mind that there are multiple approaches (and valid solutions) to some of the problems, and these approaches can differ considerably in complexity. Also, there are several tricky parts and at least one tricky problem, *so don't expect to be able to solve everything.*
- Since this is a take home exam, you will be graded on clarity and conciseness as well as accuracy and correctness. Please take the time and make the effort to make your solutions clear. Please try to use standard and simple notation, introducing only as many new symbols as is absolutely necessary. The solutions are not long, so if you find that your solution to a problem goes on and on for many pages, you should try to figure out a simpler one. I expect neat, legible exams from everyone.
- Please attach the cover page to the front of your exam. Assemble your solutions in order (problem 1, problem 2, problem 3, . . .), starting a new page for each problem.
- Please respect the honor code. Although I allow you to work on homework assignments in small groups, you cannot discuss the exam with anyone, at least until everyone has taken it.

25 points

1. *Runge-Kutta Method and Numerical Solution of ODEs*
Consider Heun's method

$$y_{n+1} = y_n + \frac{h}{2}\left[f_n + f(t_{n+1}, y_n + hf_n)\right].$$

(a) Show that Heun's method is an explicit two-stage Runge-Kutta method.

(b) Prove that Heun's method has order 2 with respect to h.

(c) Using MATLAB, draw the region of absolute stability of the method in the complex plane.

25 points

2. *Multistep Method and Numerical Solution of ODEs*
Consider the family of linear multistep methods

$$y_{n+1} = y_n + \frac{h}{2}\left[2(1-\alpha)f_{n+1} + 3\alpha f_n - \alpha f_{n-1}\right].$$

where α is a real parameter

(a) Analyze consistency and order of the methods as functions of α, determining the value α^* for which the resulting method has maximal order.

(b) Study the zero-stability of the method with $\alpha = \alpha^*$, (for a consistent multistep method, the root condition is equivalent to zero-stability).

(c) Using MATLAB, draw the region of absolute stability of the method with $\alpha = \alpha^*$ in the complex plane.

25 points

3. *Two-Point Boundary Value Problem.*
Consider the two-point boundary value problem

$$-u''(x) = f(x), \quad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

Show that

a) *Green's Function Representation*
For every $f \in C^0([0,1])$, there is a unique solution $u \in C^2([0,1])$ admitting the representation

$$u(x) = \int_0^1 G(x,s) f(s) ds,$$

where the corresponding Green's function G is defined by, for a fixed x,

$$G(x,s) = \begin{cases} s(1-x), & \text{if } 0 \le s \le x, \\ x(1-s), & \text{if } x \le s \le 1, \end{cases}$$

b) *Monotonicity Property*
If $f \in C^0([0,1])$ is a nonnegative function, then u is also nonnegative.

c) *Maximum Principle*
If $f \in C^0([0,1])$, then

$$\|u\|_\infty \le \frac{1}{8} \|f\|_\infty$$

where $\|u\|_\infty = \max_{0 \le x \le 1} |u(x)|$ is the maximum norm.

25 points

4. *Finite Difference Approximation.*

We introduce on $[0,1]$ the grid points $\{x_j\}_{j=0}^n$ given by $x_j = jh$ where $n \geq 2$ is an integer and $h = 1/n$ is the grid spacing. Let V_h be a collection of discrete functions defined at the grid points x_j for $j = 0, \cdots, n$. For $v_h \in V_h$, let $v_j = v_h(x_j)$. Let $V_h^0 = \{v_h \in V_h : v_h(0) = v_h(1) = 0\}$. For $v_h \in V_h$ we define the operator L_h by

$$(L_h v_h)(x_j) = h^{-2}\left(2v_j - v_{j-1} - v_{j+1}\right), \quad j = 1, \cdots, n-1.$$

For $f \in C^0([0,1])$, let $f_h \in V_h$ be the grid function such that $f_h(x_j) = f(x_j)$ for $j = 0, \cdots, n$. Consider the finite difference problem: find $u_h \in V_h^0$ such that

$$(L_h u_h)(x_j) = f(x_j), \quad j = 1, \cdots, n-1.$$

Show that

a) *Discrete Green's Function Representation*
The solution $u_h \in V_h^0$ has the discrete representation

$$u_h(x_j) = h \sum_{k=1}^{n-1} G(x_j, x_k) f(x_k)$$

where G is the Green's function introduced in *Two-Point Boundary Value Problem.*

b) *Discrete Monotonicity Property*
If f_h is nonnegative, then u_h is also nonnegative. (This result implies that the corresponding finite difference matrix $A = \text{tridiag}(-1, 2, -1)$ is an M-matrix.)

c) *Discrete Maximum Principle*

$$\|u_h\|_{h,\infty} \leq \frac{1}{8}\|f\|_{h,\infty}$$

where $\|u_h\|_{h,\infty} = \max_{0 \leq j \leq n} |u_h(x_j)|$ is the discrete maximum norm. (This is a stability result: the finite difference solution is bounded by the given datum f_h.)

c) *Consistency*
Assume that $f \in C^2([0,1])$. Let u be the solution of the corresponding boundary value problem. The local trunction error is the grid function τ_h defined by

$$\tau_h(x_j) = (L_h u)(x_j) - f(x_j), \quad j = 1, \cdots, n-1.$$

Then we have

$$\|\tau_h\|_{h,\infty} \leq \frac{\|f''\|_\infty}{12} h^2$$

d) *Convergence*
Assume that $f \in C^2([0,1])$. Then the nodal error $e(x_j) = u(x_j) - u_h(x_j)$ satisfies

$$\|u - u_h\|_{h,\infty} \leq \frac{\|f''\|_\infty}{96} h^2$$

i.e. u_h converges to u (in the discrete maximum norm) with second order with respect to h. (Consistency + Stability = Convergence)