1. a) A breakdown at step n ($h_{n+1,n}=0$) simplifies

$$AQ_n = Q_{n+1}\tilde{H}_n$$

i.e

$$[A]\,[q_1|\cdots|q_n] = [q_1|\cdots|q_{n+1}]\begin{bmatrix} h_{11} & \cdots & & h_{1n} \\ h_{21} & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & h_{n,n-1} & h_{nn} \\ & & & h_{n+1,n} \end{bmatrix}$$

to

$$AQ_n = Q_n H_n \qquad (*)$$

that is

$$[A]\,[q_1|\cdots|q_n] = [q_1|\cdots|q_n]\begin{bmatrix} h_{11} & \cdots & & h_{1n} \\ h_{21} & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & h_{n,n-1} & h_{nn} \end{bmatrix}$$

Then, a full $m\times m$ Hessenberg reduction

$$A = QHQ^* \qquad (**)$$

has the bloc upper-triangular structure for H

$$H = \begin{pmatrix} H_n & H_{12} \\ 0 & H_{m-n} \end{pmatrix} \qquad (***)$$

where

$$H_{m-n} = \begin{pmatrix} h_{n+1,n+1} & \cdots & & h_{n+1,m} \\ h_{n+2,n+1} & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & h_{m,m-1} & h_{m,m} \end{pmatrix}, \quad H_{12}\in\mathbb{C}^{n\times(m-n)}$$

1. b) $K_n = \langle q_1, \dots, q_n \rangle$

It follows from $(*)$ that

$$\forall i = 1, \dots, n-1, \quad A q_i = \sum_{j=1}^{i+1} h_{j,i} q_j \in K_n$$

$$A q_n = \sum_{j=1}^{n} h_{j,n} q_j \in K_n$$

Therefore

$$A K_n = \langle A q_1, \dots, A q_n \rangle \subseteq K_n$$

1. c) It follows that, from $(**)$,

$$\underset{Q}{A \sim H}, \quad \text{i.e.,} \quad \Lambda(H) = \Lambda(A)$$

(unitarly similar) set of ews.

from $(***)$, i.e. H being bloc-triangular,

$$\Lambda(H) = \Lambda(H_n) \cup \Lambda(H_{m-n}).$$

Therefore

$$\Lambda(H_n) \subseteq \Lambda(A)$$

i.e.

each ew of H_n is an ew of A.

1.d) A is non singular. Its inverse has a full 3/ Hessenberg reduction

$$A^{-1} = Q H^{-1} Q^*$$

where H^{-1} has the block upper-triangular structure

$$H^{-1} = \begin{pmatrix} H_n^{-1} & \tilde{H}_{12} \\ 0 & H_{m-n}^{-1} \end{pmatrix}, \qquad \text{with } \tilde{H}_{12} \in \mathbb{C}^{n \times (m-n)}$$

Then the solution

$$x^* = A^{-1} b = Q H^{-1} Q^* b$$

Note that $b = \|b\| q_1$.

$$Q^* b = \begin{pmatrix} q_1^* \\ \vdots \\ q_m^* \end{pmatrix} \|b\| q_1 = \|b\| \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \|b\| \underset{\in \mathbb{R}^m}{e_1}$$

$$H^{-1}(Q^* b) = \|b\| H^{-1} e_1 = \|b\| \cdot \text{first col of } H^{-1}$$

$$= \|b\| \begin{pmatrix} \text{first col of } H_n^{-1} \\ 0 \in \mathbb{R}^{m-n} \end{pmatrix} = \|b\| \begin{pmatrix} H_n^{-1} e_1 \\ 0 \end{pmatrix}$$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$

$$= \|b\| \begin{pmatrix} z \\ 0 \end{pmatrix} \quad \text{with } z = \text{first col of } H_n^{-1} = H_n^{-1} e \in \mathbb{R}^n$$

Therefore

$$x^* = Q\left[H^{-1} Q^* b\right] = \|b\| \left(Q_n, Q_{m-n}\right) \begin{pmatrix} z \\ 0 \end{pmatrix}$$

$$= \|b\| Q_n z \in \mathcal{K}_n$$

$$= \|b\| Q_n H_n^{-1} e_1, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n.$$

That is, the solution x lies in $\mathcal{K}_n$

1.b) Alternative proof.

For $\forall x \in K_n = \langle q_1, \cdots, q_n \rangle$

we have

$$x = Q_n y \quad . \quad \text{with} \quad Q_n = (q_1 | \cdots | q_n)$$

Then

$$Ax = AQ_n y = Q_n H_n y \in K_n$$

Therefore $AK_n \subseteq K_n$

1.b) Alternative proof

For $\forall x \in AK_n = \langle Aq_1 | \cdots | Aq_n \rangle$

we have

$$\begin{aligned} x &= c_1 A q_1 + \cdots + c_n A q_n \\ &= A Q_n c \quad \text{with} \quad c = (c_1, \cdots, c_n)^T \\ &= Q_n H_n c \\ &= Q_n y \quad \text{with} \quad y = H_n c \\ &\in K_n \end{aligned}$$

1 c) Alternative Proof.

Let (λ, v) be an eigenpair for H_n, i.e

$$H_n v = \lambda v.$$

From 1. a).

$$A Q_n = Q_n H_n$$

it follows that

$$A(Q_n v) = Q_n H_n v = \lambda (Q_n v)$$

So λ is an ew of A with $Q_n v$ the corresponding ev.

1. d) Alternative proof

If A is nonsingular, from 1.c) H_n is nonsingular.

The solution

$$b = \|b\| q_1, \quad q_1 = Q_n e_1$$

$$x^* = A^{-1} b = A^{-1} \|b\| q_1 = \|b\| A^{-1} Q_n e_1$$

$$= \|b\| A^{-1} (Q_n H_n) H_n^{-1} e_1 = \|b\| A^{-1} (A Q_n) H_n^{-1} e_1$$

$$= \|b\| Q_n H_n^{-1} e_1 \in K_n$$

1. d) Alternative proof.

Let $x^* = \|b\| Q_n H_n^{-1} e_1 \in K_n$

It can be shown that x^* is the solution.

$$A x^* = \|b\| (A Q_n) H_n^{-1} e_1 = \|b\| Q_n H_n H_n^{-1} e_1$$

$$= \|b\| Q_n e_1 = \|b\| q_1 = b$$

1.d) Alternative Proof:

Look for $x^* \in K_n$ s.t. $Ax^* = b$.

$$x^* = Q_n y.$$

We need to find y s.t.

$$A Q_n y = b$$

From 1.a).

$$Q_n H_n y = b$$

Since $b = \|b\| q_1 \in R(Q_n)$, we have

$$H_n y = \|b\| e_1$$

From 1.c), if A is invertible so it is H_n

and $$y = \|b\| H_n^{-1} e_1$$

Therefore $$x^* = \|b\| Q_n H_n^{-1} e_1$$

is the solution to $Ax = b$.

2)

The GMRES method at step n provides a computed solution

$$X_n = Q_n y \in K_n$$

where y minimizes the residual $\|AX_n - b\|$, i.e

$$\|A Q_n y - b\| = \text{minimum}.$$

From 1.a), if a breakdown occurs at step n,

$$A Q_n = Q_n H_n$$

then

$$\|Q_n H_n y - b\| = \text{minimum}$$

Note that $b = \|b\| q_1$, we have

$$\|H_n y - \|b\| e_1\| = \text{minimum} \quad \text{with } e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \in \mathbb{R}^n$$

which is equivalent to

$$H_n y = \|b\| e_1$$

Then $y = \|b\| H_n^{-1} e_1$

Therefore the computed solution at step n is

$$X_n = \|b\| Q_n H_n^{-1} e_1$$

which coincides with the exact solution to the system as given in 1.d)

$$X^* = \|b\| Q_n H_n^{-1} e_1$$

2/. **Alternative Proof**

The GMRES method at step n provides a computed solution $x_n \in K_n$ s.t.

$$(*)\quad \|Ax_n - b\| = \min_{x \in K_n} \|Ax - b\|$$

From 1.d), if a breakdown occurs at step n, the exact solution x^* to $Ax = b$ lies in K_n,

i.e, $x^* \in K_n$.

It follows from $(*)$ that

$$0 \le \|Ax_n - b\| \le \|Ax^* - b\| \le 0$$

which implies

$$x_n = x^*.$$

3. a). First we show that

$$\alpha_n = \frac{r_{n-1}^T r_{n-1}}{p_{n-1}^T A p_{n-1}}.$$

It suffices to show that

$$r_{n-1}^T r_{n-1} = p_{n-1}^T r_{n-1}. \quad (*)$$

Note that $p_{n-1} = r_{n-1} + \beta_{n-1} p_{n-2}$

Then $p_{n-1}^T r_{n-1} = (r_{n-1}^T + \beta_{n-1} p_{n-2}^T) r_{n-1}$

$$= r_{n-1}^T r_{n-1} + \beta_{n-1} p_{n-2}^T r_{n-1}$$

Since $r_{n-1} \perp K_{n-1}$ and $p_{n-2} \in K_{n-1}$,

$$p_{n-2}^T r_{n-1} = 0.$$

Then $p_{n-1}^T r_{n-1} = r_{n-1}^T r_{n-1}$

So the result follows.

Next we show that

$$\beta_n = \frac{r_n^T r_n}{r_{n-1}^T r_{n-1}}$$

Note that $r_n = r_{n-1} - \alpha_n A p_{n-1}$

i.e $\quad -A p_{n-1} = \dfrac{r_n - r_{n-1}}{\alpha_n} \quad (**)$

Then $\beta_n = - \dfrac{r_n^T (-A p_{n-1})}{p_{n-1}^T (-A p_{n-1})} = - \dfrac{r_n^T \left(\frac{r_n - r_{n-1}}{\alpha_n} \right)}{p_{n-1}^T \left(\frac{r_n - r_{n-1}}{\alpha_n} \right)}$

$$= \frac{r_n^T r_n - r_n^T r_{n-1}}{p_{n-1}^T r_{n-1} - p_{n-1}^T r_n}$$

Note that

$$r_n \perp K_n, \quad r_{n-1} \in K_n, \quad r_n^T r_{n-1} = 0$$
$$p_{n-1} \in K_n, \quad r_n^T p_{n-1} = 0$$

And as proved above, (*),

$$p_{n-1}^T r_{n-1} = r_{n-1}^T r_{n-1}$$

We have

$$\beta_n = \frac{r_n^T r_n}{r_{n-1}^T r_{n-1}}.$$

3. b).

$$\text{RHS} = -\frac{1}{\alpha_{n+1}} r_{n+1} + \left(\frac{1}{\alpha_{n+1}} + \frac{\beta_n}{\alpha_n}\right) r_n - \frac{\beta_n}{\alpha_n} r_{n-1}$$

$$= -\frac{r_{n+1} - r_n}{\alpha_{n+1}} - \beta_n \left(-\frac{r_n - r_{n-1}}{\alpha_n} \right)$$

It follows from **) in 3. a).

$$\text{RHS} = A p_n - \beta_n A p_{n-1}$$

$$= A \left(p_n - \beta_n p_{n-1} \right)$$

$$= A r_n = \text{LHS}.$$