Problems and Comments on Induction
Chapter 5

Section 5.1, Problems: 25, 32, 35, 51

Comments. We will take the following for granted: Let S be a non-empty subset of natural numbers. Then S contains a smallest element. This is called the *well-ordering principle*. The argument for showing this principle is clear. Let n be any element in $S.$ Because S is non-empty, there must be such an $n.$ If n is already the smallest element in S, we are done. Otherwise, there is a smaller element n_1 in $S.$ If n_1 is the smallest element in $S,$ we are done. Otherwise there is a smaller element n_2 in $S.$ Because we cannot have an infinite descending chain $n > n_1 > n_2 > n_3 > \cdots$ of natural numbers smaller than $n,$ we must arrive this way at the smallest number in $S.$

From the well-ordering principle we can deduce the proof principle of *Mathematical Induction*. In order to prove a statement about natural numbers, $P(n),$ it is enough to prove $P(0),$ which is the **basis step**, together with the **inductive step**, which is the implication $P(n) \to P(n+1).$ Indeed, if we had some n for which P would not be true, then the set $S = \{n | \neg P(n)\}$ would be non-empty. Thus S would have a least element, $m.$ This m cannot be 1, because P is true for $1.$ Thus m must have a predecessor, $m-1,$ which is a natural number. But $P(m-1)$ is true. We have already chosen as number m the smallest number for which P is not true, and $m-1$ is smaller than $m.$ But then the inductive step: $P(m-1) \to P(m)$ yields that $P(m)$ must hold. But this is a contradiction, P does not hold for $m.$

Example 11, p. 247, is a beautiful and non-trivial example of mathematical induction.

There is a second version of induction. Assume that we can show the following: $P(1)$ holds and $P(n)$ holds, *in case that* $P(k)$ holds *for every* $k < n.$ Then P holds for all natural numbers $n.$ Indeed, assume that we had a number n for which P does not hold. We take the smallest such number, $n.$ It cannot be 1. But by the choice of $m,$ we have $P(k)$ for all $k < n.$ But then $P(n)$ holds, which is a contradiction.

This second principle of complete induction is often used in algebra. For example in order to show that every natural number is a product of primes. We define 1 as the empty product of primes. Then, if n is any natural number, it is either a prime, and we are done, or it is the product of two smaller numbers n_1 and $n_2.$ Assuming that every number smaller than n is a product of primes, n_1 as well as n_2 are products of primes. But then $n = n_1 \cdot n_2$ is a product of primes.