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A FINITE ELEMENT METHOD FOR ELLIPTIC EQUATIONS ON SURFACES*

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Abstract. In this paper a new finite element approach for the discretization of elliptic partial differential equations on surfaces is treated. The main idea is to use finite element spaces that are induced by triangulations of an "outer" domain to discretize the partial differential equation on the surface. The method is particularly suitable for problems in which there is a coupling with a flow problem in an outer domain that contains the surface. We give an analysis that shows that the method has optimal order of convergence both in the H^1- and in the L^2-norm. Results of numerical experiments are included that confirm this optimality.

Key words. surface, interface, finite element, level set method, two-phase flow, Marangoni

AMS subject classifications. 58J32, 65N15, 65N30, 76D45, 76T99

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1. Introduction.

Moving hypersurfaces and interfaces appear in many physical processes, for example, in multiphase flows and flows with free surfaces. Certain mathematical models involve elliptic partial differential equations posed *on* such surfaces. This happens, for example, in multiphase fluids if one takes so-called surface active agents (surfactants) into account. These surfactants induce tangential surface tension forces and thus cause Marangoni phenomena [9, 10]. Numerical simulations play an important role in a better understanding and prediction of processes involving this or other surface phenomena. In mathematical models surface equations are often coupled with other equations that are formulated in a (fixed) domain which contains the surface. In such a setting a common approach is to use a splitting scheme that allows to solve at each time step a sequence of simpler (decoupled) equations. In doing so one has to solve numerically at each time step an elliptic type of equation on a surface. The surface may vary from one time step to another and usually only some discrete approximation of the surface is available. A well-known finite element method for solving elliptic equations on surfaces, initiated by the paper [5], consists of approximating the surface by a piecewise polygonal surface and using a finite element space on a triangulation of this discrete surface, cf. [3, 9]. If the surface is changing in time, then this approach leads to time-dependent triangulations and time-dependent finite element spaces. Implementing this requires substantial data handling and programming effort. Another approach has recently been introduced in [2]. The method in that paper applies to cases in which the surface is given implicitly by some level set function, and the key idea is to solve the partial differential equation on a narrow band around the surface. Unfitted finite element spaces on this narrow band are used for discretization.

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In this paper we introduce a new technique for the numerical solution of an elliptic equation posed on a hypersurface. The main idea is to use time-*in*dependent finite element spaces that are induced by triangulations of an "outer" domain to discretize the partial differential equation on the surface. Our method is particularly suitable for problems in which the surface is given implicitly by a level set or VOF function and in which there is a coupling with a flow problem in a fixed outer domain. If in such problems one uses finite element techniques for the discretization of the flow equations in the outer domain, this setting immediately results in an easy to implement discretization method for the surface equation. The new approach does not require additional surface elements. If the surface varies in time, one has to recompute the surface stiffness matrix using the same data structures each time. Moreover, quadrature routines that are needed for these computations are often available already, since they are needed in other surface related calculations, for example, surface tension forces. Opposite to the method in [2] we do not use an *extension* of the surface partial differential equation but instead use a *restriction* of the outer finite element spaces.

We prove that the method has optimal order of convergence in H^1- and L^2-norms. The analysis requires shape regularity of the outer triangulation, but *does not* require any type of shape regularity for discrete surface elements. The number of unknowns in the resulting algebraic systems is almost the same as in the approach based on the surface finite element spaces. All these properties make the new method very attractive both from the theoretical and the practical (implementation) point of view.

Although our primal objective is to efficiently solve equations on moving and implicitly defined surfaces, the method is also well suited for problems with steady and/or explicitly given surfaces.

The remainder of the paper is organized as follows. In section 2 we present the finite element method for the model example of the Laplace–Beltrami equation. Section 3 contains the main theoretical results of the paper concerning the approximation properties of the finite element spaces and discretization error bounds for the new method. Finally, in section 4 results of numerical experiments are given, which support the theoretical analysis of the paper.

2. Laplace–Beltrami equation and finite element discretization. In applications, the finite element method that is presented in this section is particularly suited for discretization of elliptic equations on a *moving* manifold $\Gamma = \Gamma(t)$. In this paper, however, we restrict ourselves to the case of a fixed sufficiently smooth manifold Γ ($= \Gamma(t_n)$) without boundary. As a model problem for an elliptic equation we consider the pure diffusion (i.e., Laplace–Beltrami) equation.

We assume that Ω is an open subset in $\mathbb{R}^3$ and Γ a connected C^2 compact hypersurface contained in Ω. For a sufficiently smooth function $g : \Omega \to \mathbb{R}$ the tangential derivative (along Γ) is defined by

$$\nabla_\Gamma g = \nabla g - \nabla g \cdot \mathbf{n}_\Gamma \, \mathbf{n}_\Gamma. \tag{2.1}$$

By Δ_Γ we denote the *Laplace–Beltrami operator* on Γ. We consider the Laplace–Beltrami problem in weak form: For given $f \in L^2(\Gamma)$ with $\int_\Gamma f \mathrm{d}\mathbf{s} = 0$, determine $u \in H^1(\Gamma)$, with $\int_\Gamma u \, \mathrm{d}\mathbf{s} = 0$, such that

$$\int_\Gamma \nabla_\Gamma u \nabla_\Gamma v \, \mathrm{d}\mathbf{s} = \int_\Gamma f v \, \mathrm{d}\mathbf{s} \qquad \text{for all} \;\; v \in H^1(\Gamma). \tag{2.2}$$

The solution u is unique and satisfies $u \in H^2(\Gamma)$ with $\|u\|_{H^2(\Gamma)} \leq c \|f\|_{L^2(\Gamma)}$ and a constant c independent of f, cf. [5].

For the discretization of this problem one needs an approximation Γ_h of Γ. We assume that this approximate manifold is constructed as follows. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of tetrahedral triangulations of a *fixed* domain $\Omega \subset \mathbb{R}^3$ that contains Γ. These triangulations are assumed to be regular, consistent, and stable [1]. Take $\mathcal{T}_h \in \{\mathcal{T}_h\}_{h>0}$. We assume that Γ_h is a $C^{0,1}$ surface without a boundary and Γ_h can be partitioned in planar segments, triangles or quadrilaterals, consistent with the outer triangulation $\mathcal{T}_h$. This can be formally defined as follows. For any tetrahedron $S_T \in \mathcal{T}_h$ such that $\text{meas}_2(S_T \cap \Gamma_h) > 0$ define $T = S_T \cap \Gamma_h$. We assume that each T is *planar*, i.e., either a triangle or a quadrilateral. Thus, Γ_h can be decomposed as

$$\Gamma_h = \cup_{T \in \mathcal{F}_h} T, \tag{2.3}$$

where $\mathcal{F}_h$ is the set of all triangles or quadrilaterals T such that $T = S_T \cap \Gamma_h$ for some tetrahedron $S_T \in \mathcal{T}_h$. Note that if T coincides with a face of an element in $\mathcal{T}_h$ then the corresponding S_T is not unique. In this case, we choose one arbitrary but fixed tetrahedron S_T, which has T as a face.

Remark 1. We briefly explain an approach for the construction of an approximation Γ_h of Γ that is used in our applications in two-phase flow problems, cf. [6, 8, 7]. The interface Γ is represented as the zero level of a (unknown) level set function ϕ. The level set equation for ϕ is discretized with continuous piecewise quadratic finite elements on the tetrahedral triangulation $\mathcal{T}_h$. The use of piecewise *quadratics* (instead of piecewise linears) allows an accurate discretization of the surface tension force (which depends on the curvature of Γ). The (given) piecewise quadratic finite element approximation of ϕ on $\mathcal{T}_h$ is denoted by ϕ_h. We now introduce one further regular refinement of $\mathcal{T}_h$, resulting in $\mathcal{T}'_h = \mathcal{T}_{\frac{h}{2}}$. Let $I(\phi_h)$ be the continuous piecewise *linear* function on $\mathcal{T}'_h$, which interpolates ϕ_h at all vertices of all tetrahedra in $\mathcal{T}'_h$. The approximation of the interface Γ is defined by

$$\Gamma_h := \{\, \mathbf{x} \in \Omega \mid I(\phi_h)(\mathbf{x}) = 0 \,\} \tag{2.4}$$

and consists of piecewise planar segments. The mesh size parameter h is the maximal diameter of these segments. This maximal diameter is approximately the maximal diameter of the tetrahedra in $\mathcal{T}'_h$ that contain the discrete interface, i.e., $h = h_\Gamma$ is approximately the maximal diameter of the tetrahedra in $\mathcal{T}'_h$ that are close to the interface. In Figure 2.1 we illustrate this construction for the two-dimensional (2D) case.

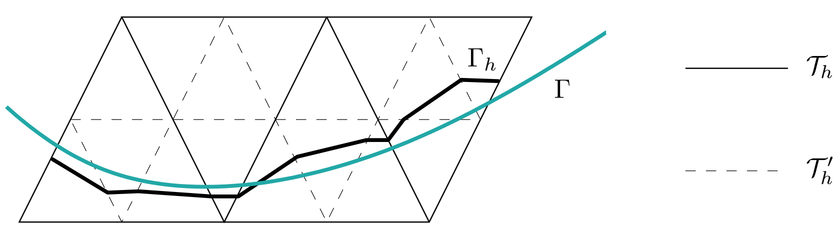

FIG. 2.1. *Construction of approximate interface for the 2D case.*

Each of the planar segments of Γ_h is either a triangle or a quadrilateral. This construction of Γ_h satisfies the assumptions made above. It can be shown that under reasonable assumption, as explained in Remark 7 below, the approximation Γ_h is "close to" Γ in the following sense (cf. (3.14), (3.15)): $\text{dist}(\Gamma_h, \Gamma) \leq c_0 h^2$, and $\text{ess sup}_{\mathbf{x}\in\Gamma_h} \|\mathbf{n}(\mathbf{x}) - \mathbf{n}_h(\mathbf{x})\| \leq \tilde{c}_0 h$, where $\mathbf{n}$ is the extension of $\mathbf{n}_\Gamma$ in a neighborhood of Γ and $\mathbf{n}_h$ is a unit normal on Γ_h. In Figure 2.2 we show a part of Γ_h that is constructed as explained above for a two-phase flow application with a rising droplet.

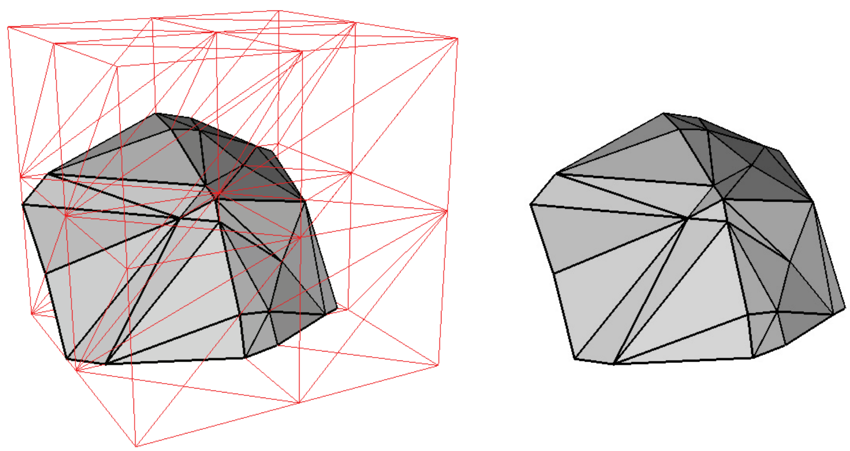

FIG. 2.2. *Example of a part of* Γ_h *in a two-phase flow application.*

The main new idea of this paper is that for discretization of the problem (2.2) *we use a finite element space induced by the continuous linear finite elements on* $\mathcal{T}_h$. This is done as follows. We define a subdomain that contains Γ_h:

$$\omega_h := \cup_{T\in\mathcal{F}_h} S_T. \tag{2.5}$$

We introduce the finite element space

$$V_h := \{\, v_h \in C(\omega_h) \mid v_{|S_T} \in P_1 \ \text{ for all } \ T \in \mathcal{F}_h \,\}, \tag{2.6}$$

where P_1 is the space of polynomials of degree one. The space V_h induces the following space on Γ_h:

$$V_h^\Gamma := \{\psi_h \in H^1(\Gamma_h) \,|\, \exists\; v_h \in V_h :\; \psi_h = v_h|_{\Gamma_h} \}. \tag{2.7}$$

This space is used for a Galerkin discretization of (2.2) as follows: Determine $u_h \in V_h^\Gamma$, with $\int_{\Gamma_h} u_h \mathrm{d}\mathbf{s}_h = 0$, such that

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \nabla_{\Gamma_h} \psi_h \, \mathrm{d}\mathbf{s}_h = \int_{\Gamma_h} f_h \psi_h \, \mathrm{d}\mathbf{s}_h \qquad \text{for all } \ \psi_h \in V_h^\Gamma, \tag{2.8}$$

with f_h an extension of f such that $\int_{\Gamma_h} f_h \mathrm{d}\mathbf{s}_h = 0$, cf. section 3.3. Due to the Lax–Milgram lemma this problem has a unique solution u_h. In section 3 we present a discretization error analysis of this method that shows that under reasonable assumptions we have optimal error bounds. In section 4 we show results of numerical experiments that confirm the theoretical analysis. As far as we know this method for discretization of a partial differential equation on a surface is new. In the remarks below we give some comments related to this approach.

Remark 2. The family $\{\mathcal{T}_h\}_{h>0}$ is *shape-regular*, but the family $\{\mathcal{F}_h\}_{h>0}$ in general is *not shape-regular*. In our numerical experiments (cf. section 4) $\mathcal{F}_h$ contains a significant number of strongly deteriorated triangles that have very small angles. Moreover, neighboring triangles can have very different areas, cf. Figure 4.1. As we will prove in section 3, optimal discretization bounds hold if $\{\mathcal{T}_h\}_{h>0}$ is shape-regular; for $\{\mathcal{F}_h\}_{h>0}$, shape-regularity is *not* required.

Remark 3. Let $(\xi_i)_{1\le i\le m}$ be the collection of all vertices of all tetrahedra in ω_h and ϕ_i the nodal linear finite element basis function corresponding to ξ_i. Then V_h^Γ is spanned by the functions $\phi_i|_{\Gamma_h}$, $1 \le i \le m$. These functions, however, are *not* necessarily independent. In computations we use this generating system $\phi_i|_{\Gamma_h}$, $1 \le i \le m$ for solving the discrete problem (2.8). Properties that are of interest for the numerical solution of the resulting linear system, such as conditioning of the mass and stiffness matrix, are analyzed in the forthcoming paper [11].

Remark 4. In the implementation of this method one has to compute integrals of the form

$$\int_T \nabla_{\Gamma_h}\phi_j \nabla_{\Gamma_h}\phi_i \,\mathrm{d}\mathbf{s}, \qquad \int_T f_h\phi_i \,\mathrm{d}\mathbf{s} \quad \text{for } \ T \in \mathcal{F}_h.$$

The domain T is either a triangle or a quadrilateral. The first integral can be computed exactly. For the second one standard quadrature rules can be applied.

Remark 5. Each quadrilateral in $\mathcal{F}_h$ can be subdivided into two triangles. Let $\tilde{\mathcal{F}}_h$ be the induced set consisting of *only* triangles such that $\cup_{T\in\tilde{\mathcal{F}}_h} T = \Gamma_h$. Define

$$W_h^\Gamma := \{\, \psi_h \in C(\Gamma_h) \mid \psi_h|_T \in P_1 \quad \text{for all } \ T \in \tilde{\mathcal{F}}_h \,\}. \tag{2.9}$$

The space W_h^Γ is the space of continuous functions that are piecewise linear on the triangles of Γ_h. Clearly, $V_h^\Gamma \subset W_h^\Gamma$ holds. There are, however, situations in which $V_h^\Gamma \ne W_h^\Gamma$. A 2D illustration of this is given in Figure 2.3.

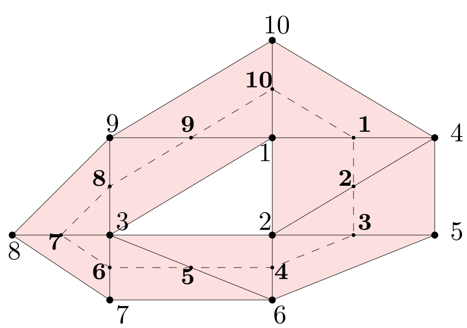

FIG. 2.3. *Example of* $V_h^\Gamma \ne W_h^\Gamma$.

In this example ω_h consists of 10 triangles (shaded). The nodal basis functions corresponding to these basis functions are denoted by $\{\phi_i\}_{1\le i\le 10}$. The line segments of the interface Γ_h (denoted by - -) intersect midpoints of edges of the triangles. The space W_h^Γ consists of piecewise linears on Γ_h and is spanned by the one-dimensional (1D) nodal basis functions at the intersection points labeled by boldface $\mathbf{1}, \ldots, \mathbf{10}$. Clearly, $\dim(W_h^\Gamma) = 10$. In this example we have $\dim(V_h^\Gamma) = 9$. For the piecewise linear function $v = \sum_{i=1}^{10} \alpha_i\phi_i$, with $\alpha_i = -1$ for $i = 1, 2, 3$ and $\alpha_i = 1$ for $i = 4, \ldots, 10$, we have $v_{|\Gamma_h} = 0$.

The example in Remark 5 shows that the finite element space V_h^Γ can be smaller than W_h^Γ, and therefore approximation properties of V_h^Γ do not follow directly from those of W_h^Γ. Moreover, the triangulations $\{\tilde{\mathcal{F}}_h\}_{h>0}$ of Γ_h are not shape regular, cf. Remark 2 and Figure 4.1. Thus, it is not clear how (optimal) approximation error bounds for the standard linear finite element space W_h^Γ in (2.9) can be derived.

3. Discretization error analysis. In this section we derive discretization error bounds, both in the H^1- and the L^2-norm on Γ_h. We first collect some preliminaries in section 3.1, then derive approximation error bounds in section 3.2, and finally present discretization error bounds in section 3.3.

3.1. Preliminaries. We will need a Poincare type inequality that is given in the following lemma.

LEMMA 3.1. *Consider a bounded domain $\Omega \subset \mathbb{R}^n$ and a subdomain $S \subset \Omega$. Assume that Ω is such that the Neumann–Poincare inequality is valid:*

$$\|f\|_{L^2(\Omega)} \le C_P \,\|\nabla f\|_{L^2(\Omega)} \quad \textit{for all} \;\; f \in H^1(\Omega) \;\; \textit{with} \;\; \int_\Omega f \, d\mathbf{x} = 0. \tag{3.1}$$

Then for any $f \in H^1(\Omega)$ the following estimate holds:

$$\|f\|^2_{L^2(\Omega)} \le \frac{|\Omega|}{|S|}\left(2\|f\|^2_{L^2(S)} + 3C_P^2\|\nabla f\|^2_{L^2(\Omega)}\right). \tag{3.2}$$

Proof. The proof uses a technique developed by Sobolev ([13], Ch. I) for building equivalent norms on $W_q^l(\Omega)$ (Sobolev spaces). We consider the simple case with $q = 2$, $l = 1$, i.e., $H^1(\Omega)$. We introduce the projectors $\Pi_k : H^1(\Omega) \to \mathbb{R}$, $k = 1, 2$:

$$\Pi_1 f := |\Omega|^{-1}\int_\Omega f \, d\mathbf{x}, \qquad \Pi_2 f := |S|^{-1}\int_S f \, d\mathbf{x}.$$

Since $\|(I - \Pi_1)f\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega)} - |\Omega||\Pi_1 f|^2$, the Neumann–Poincare inequality (3.1) can be rewritten in the equivalent form:

$$\|f\|^2_{L^2(\Omega)} \le |\Omega||\Pi_1 f|^2 + C_P^2\|\nabla f\|^2_{L^2(\Omega)} \quad \text{for all} \;\; f \in H^1(\Omega). \tag{3.3}$$

For any $f \in H^1(\Omega)$, with $\Pi_1 f = 0$, the Cauchy and Neumann–Poincare inequality implies

$$\begin{aligned} |\Pi_2 f| = |S|^{-1}\left|\int_S f \, d\mathbf{x}\right| &\le |S|^{-\frac{1}{2}}\|f\|_{L^2(S)} \\ &\le |S|^{-\frac{1}{2}}\|f\|_{L^2(\Omega)} \le C_p|S|^{-\frac{1}{2}}\|\nabla f\|_{L^2(\Omega)}. \end{aligned} \tag{3.4}$$

Define $M := C_P|S|^{-\frac{1}{2}}$. Note that for $f \in H^1(\Omega)$ we have $\Pi_1(I - \Pi_1)f = 0$ and thus from (3.4) we obtain:

$$|(\Pi_2 - \Pi_1)f| = |\Pi_2(I - \Pi_1)f| \le M\|\nabla(I - \Pi_1)f\|_{L^2(\Omega)} = M\|\nabla f\|_{L^2(\Omega)}.$$

Hence, for any $f \in H^1(\Omega)$ we have

$$\begin{aligned} |\Pi_1 f|^2 + M^2\|\nabla f\|^2_{L^2(\Omega)} &\le 2|\Pi_2 f|^2 + 2|(\Pi_2 - \Pi_1)f|^2 + M^2\|\nabla f\|^2_{L^2(\Omega)} \\ &\le 2|\Pi_2 f|^2 + 3M^2\|\nabla f\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.5}$$

Estimates (3.3) and (3.5) imply:

$$\begin{aligned}\|f\|^2_{L^2(\Omega)} &\le \max\{|\Omega|, C_P^2 M^{-2}\}\left(|\Pi_1 f|^2 + M^2\|\nabla f\|^2_{L^2(\Omega)}\right)\\ &= |\Omega|\left(|\Pi_1 f|^2 + M^2\|\nabla f\|^2_{L^2(\Omega)}\right)\\ &\le |\Omega|\left(2|\Pi_2 f|^2 + 3M^2\|\nabla f\|^2_{L^2(\Omega)}\right)\\ &\le |\Omega|\left(2|S|^{-1}\|f\|^2_{L^2(S)} + 3M^2\|\nabla f\|^2_{L^2(\Omega)}\right)\\ &= |\Omega||S|^{-1}\left(2\|f\|^2_{L^2(S)} + 3C_P^2\|\nabla f\|^2_{L^2(\Omega)}\right),\end{aligned}$$

which proves the inequality in (3.2). □

Remark 6. In the analysis below we shall apply Lemma 3.1 for the case of *convex* domain Ω. For convex domains the following upper bound is well known [12] for the Poincare constant:

$$C_P \le \frac{\operatorname{diam}(\Omega)}{\pi}. \tag{3.6}$$

We define a neighborhood of Γ:

$$U = \{\, \mathbf{x} \in \mathbb{R}^3 \mid \operatorname{dist}(\mathbf{x}, \Gamma) < c\,\},$$

with c sufficiently small and assume that $\Gamma_h \subset U$. Let $d : U \to \mathbb{R}$ be the signed distance function, $|d(x)| := \operatorname{dist}(\mathbf{x}, \Gamma)$ for all $\mathbf{x} \in U$. Thus, Γ is the zero level set of d. We assume $d < 0$ on the interior of Γ and $d > 0$ on the exterior. Note that $\mathbf{n}_\Gamma = \nabla d$ on Γ. We define $\mathbf{n}(\mathbf{x}) := \nabla d(\mathbf{x})$ for all $\mathbf{x} \in U$. Thus, $\mathbf{n} = \mathbf{n}_\Gamma$ on Γ and $\|\mathbf{n}(\mathbf{x})\| = 1$ for all $\mathbf{x} \in U$. Here and in the remainder $\|\cdot\|$ denotes the Euclidean norm. The Hessian of d is denoted by $\mathbf{H}$:

$$\mathbf{H}(\mathbf{x}) = D^2 d(\mathbf{x}) \in \mathbb{R}^{3\times 3} \quad \text{for all} \;\; \mathbf{x} \in U. \tag{3.7}$$

The eigenvalues of $\mathbf{H}(\mathbf{x})$ are denoted by $\kappa_1(\mathbf{x}), \kappa_2(\mathbf{x})$, and 0. For $\mathbf{x} \in \Gamma$ the eigenvalues $\kappa_i(\mathbf{x})$, $i = 1, 2$, are the principal curvatures.

We will need the orthogonal projection

$$\mathbf{P}(\mathbf{x}) = \mathbf{I} - \mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x})^T \quad \text{for} \;\; \mathbf{x} \in U.$$

Note that the tangential derivative can be written as $\nabla_\Gamma g(\mathbf{x}) = \mathbf{P}\nabla g(\mathbf{x})$ for $\mathbf{x} \in \Gamma$. We introduce a locally orthogonal coordinate system by using the projection $\mathbf{p} : U \to \Gamma$:

$$\mathbf{p}(\mathbf{x}) = \mathbf{x} - d(\mathbf{x})\mathbf{n}(\mathbf{x}) \quad \text{for all} \;\; \mathbf{x} \in U.$$

We assume that the decomposition $\mathbf{x} = \mathbf{p}(\mathbf{x}) + d(\mathbf{x})\mathbf{n}(\mathbf{x})$ is unique for all $\mathbf{x} \in U$. Note that

$$\mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{p}(\mathbf{x})) \quad \text{for all} \;\; \mathbf{x} \in U.$$

We use an extension operator defined as follows. For a function v on Γ we define

$$v^e(\mathbf{x}) := v(\mathbf{x} - d(\mathbf{x})\mathbf{n}(\mathbf{x})) = v(\mathbf{p}(\mathbf{x})) \quad \text{for all} \;\; \mathbf{x} \in U,$$

i.e., v is extended along normals on Γ. We define a discrete analogon of the orthogonal projection $\mathbf{P}$:

$$\mathbf{P}_h(\mathbf{x}) := \mathbf{I} - \mathbf{n}_h(\mathbf{x})\mathbf{n}_h(\mathbf{x})^T \quad \text{for} \;\; \mathbf{x} \in \Gamma_h, \;\; \mathbf{x} \text{ not on an edge.}$$

Here $\mathbf{n}_h(\mathbf{x})$ denotes the (outward pointing) normal at $\mathbf{x} \in \Gamma_h$ ($\mathbf{x}$ not on an edge). The tangential derivative along Γ_h can be written as $\nabla_{\Gamma_h} g(\mathbf{x}) = \mathbf{P}_h(\mathbf{x})\nabla g(\mathbf{x})$ for $\mathbf{x} \in \Gamma_h$ (not on an edge).

In the analysis we use techniques from [3, 5]. For example, the formula

$$\nabla u^e(\mathbf{x}) = (\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x}))\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) \quad \text{a.e. on} \;\; U \tag{3.8}$$

(cf. section 2.3 in [3]), which implies

$$\nabla_{\Gamma_h} v^e(\mathbf{x}) = \mathbf{P}_h(\mathbf{x})\big(\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x})\big)\nabla_\Gamma v(\mathbf{p}(\mathbf{x})) \quad \text{a.e. on} \;\; \Gamma_h. \tag{3.9}$$

Furthermore, for u sufficiently smooth and $|\mu| = 2$, the inequality

$$|D^\mu u^e(\mathbf{x})| \le c \left(\sum_{|\mu|=2} |\mathrm{D}^\mu_\Gamma u(\mathbf{p}(\mathbf{x}))| + \|\nabla_\Gamma u(\mathbf{p}(\mathbf{x}))\| \right) \quad \text{a.e. on} \;\; U \tag{3.10}$$

holds, cf. lemma 3 in [5]. We define an h-neighborhood of Γ:

$$U_h = \{\, \mathbf{x} \in \mathbb{R}^3 \mid \mathrm{dist}(\mathbf{x}, \Gamma) < c_1 h \,\}$$

and assume that h is sufficiently small, such that $\omega_h \subset U_h \subset U$ and

$$5c_1 h < \left(\max_{i=1,2} \|\kappa_i\|_{L^\infty(\Gamma)} \right)^{-1}. \tag{3.11}$$

From (2.5) in [3] we have the following formula for the principal curvatures κ_i:

$$\kappa_i(\mathbf{x}) = \frac{\kappa_i(\mathbf{p}(\mathbf{x}))}{1 + d(\mathbf{x})\kappa_i(\mathbf{p}(\mathbf{x}))} \quad \text{for } \mathbf{x} \in U. \tag{3.12}$$

Hence, from (3.11) and (3.12) it follows that

$$\|d\|_{L^\infty(U_h)} \max_{i=1,2} \|\kappa_i\|_{L^\infty(U_h)} \le \frac{1}{4} \tag{3.13}$$

holds. In the remainder we assume that

$$\text{ess sup}_{\mathbf{x}\in\Gamma_h} |d(\mathbf{x})| \le c_0 h^2, \tag{3.14}$$

$$\text{ess sup}_{\mathbf{x}\in\Gamma_h} \|\mathbf{n}(\mathbf{x}) - \mathbf{n}_h(\mathbf{x})\| \le \tilde{c}_0 h \tag{3.15}$$

holds.

Remark 7. Related to the assumptions in (3.14)–(3.15) we note the following. Consider an approach as outlined in Remark 1 in which the approximation Γ_h of Γ is constructed using a level set method and a piecewise quadratic finite approximation ϕ_h of the level set function ϕ. We assume that the level set function ϕ equals the signed distance function d, i.e., $\phi = d$ and that for the finite element approximation an error bound

$$\|\phi_h - \phi\|_{L^\infty(\omega_h)} + h\|\phi_h - \phi\|_{H^{1,\infty}(\omega_h)} \le ch^k \|\phi\|_{H^{k,\infty}(\omega_h)}, \quad k = 1, 2, \tag{3.16}$$

holds (which is reasonable for the case of piecewise quadratics and if ϕ is sufficiently smooth). Let I be the nodal interpolation operator on the vertices of the triangulation ω_h. Using standard properties of this operator and the error bound in (3.16) one obtains

$$\begin{aligned}\|I\phi_h - \phi\|_{L^\infty(\omega_h)} &\leq \|I(\phi_h - \phi)\|_{L^\infty(\omega_h)} + \|I\phi - \phi\|_{L^\infty(\omega_h)} \\ &\leq \|\phi_h - \phi\|_{L^\infty(\omega_h)} + ch^2\|\phi\|_{H^{2,\infty}(\omega_h)} \\ &\leq c\,h^2\|\phi\|_{H^{2,\infty}(\omega_h)},\end{aligned}$$

and thus for $\mathbf{x} \in \Gamma_h$ we have $|d(\mathbf{x})| = |I(\phi_h)(\mathbf{x}) - \phi(\mathbf{x})| \leq c\,h^2$ and hence (3.14) is satisfied.

We also have

$$\begin{aligned}\|I\phi_h - \phi\|_{H^{1,\infty}(\omega_h)} &\leq \|I(\phi_h - \phi)\|_{H^{1,\infty}(\omega_h)} + \|I\phi - \phi\|_{H^{1,\infty}(\omega_h)} \\ &\leq c\,\|\phi_h - \phi\|_{H^{1,\infty}(\omega_h)} + ch\|\phi\|_{H^{2,\infty}(\omega_h)} \leq ch\|\phi\|_{H^{2,\infty}(\omega_h)}.\end{aligned}$$

Using this and $\|\nabla\phi\| = 1$ we then have $\|\nabla I(\phi_h)(\mathbf{x})\| = 1 + \mathcal{O}(h)$ for $\mathbf{x} \in \Gamma_h$. For $\mathbf{x} \in \Gamma_h$ (not on an edge) we obtain

$$\begin{aligned}\|\mathbf{n}_h(\mathbf{x}) - \mathbf{n}(\mathbf{x})\| &= \left\| \frac{\nabla I(\phi_h)(x)}{\|\nabla I(\phi_h)(\mathbf{x})\|} - \nabla\phi(\mathbf{x}) \right\| \\ &\leq \left| \frac{1}{\|\nabla I(\phi_h)(\mathbf{x})\|} - 1 \right| \cdot \|\nabla I(\phi_h)(\mathbf{x})\| + \|\nabla I(\phi_h)(\mathbf{x}) - \nabla\phi(\mathbf{x})\| \leq c\,h,\end{aligned}$$

and thus (3.15) is satisfied (for h sufficiently small).

LEMMA 3.2. *There are constants $c_1 > 0$ and c_2 independent of h such that for all $u \in H^2(\Gamma)$ the following inequalities hold:*

$$c_1\|u^e\|_{L^2(U_h)} \leq \sqrt{h}\|u\|_{L^2(\Gamma)} \leq c_2\|u^e\|_{L^2(U_h)}, \tag{3.17}$$

$$c_1\|\nabla u^e\|_{L^2(U_h)} \leq \sqrt{h}\|\nabla_\Gamma u\|_{L^2(\Gamma)} \leq c_2\|\nabla u^e\|_{L^2(U_h)}, \tag{3.18}$$

$$\|D^\mu u^e\|_{L^2(U_h)} \leq c_2\sqrt{h}\|u\|_{H^2(\Gamma)}, \quad |\mu| = 2. \tag{3.19}$$

Proof. Note that $u \in H^2(\Gamma)$ is continuous and thus u^e is well defined. Define

$$\mu(\mathbf{x}) := \big(1 - d(\mathbf{x})\kappa_1(\mathbf{x})\big)\big(1 - d(\mathbf{x})\kappa_2(\mathbf{x})\big), \quad \mathbf{x} \in U_h.$$

From (2.20), (2.23) in [3] we have

$$\mu(\mathbf{x})\mathrm{d}\mathbf{x} = \mathrm{d}r\mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x})), \quad \mathbf{x} \in U,$$

where $\mathrm{d}\mathbf{x}$ is the measure in U_h, $\mathrm{d}\mathbf{s}$ the surface measure on Γ, and r the local coordinate at $\mathbf{x} \in \Gamma$ in the direction $\mathbf{n}(\mathbf{p}(\mathbf{x})) = \mathbf{n}(\mathbf{x})$. Using (3.13) we get

$$\frac{9}{16} \leq \mu(\mathbf{x}) \leq \frac{25}{16} \quad \text{for all } \ \mathbf{x} \in U_h. \tag{3.20}$$

Using the local coordinate representation $\mathbf{x} = (\mathbf{p}(\mathbf{x}), r)$, for $\mathbf{x} \in U$, we have

$$\begin{aligned}\int_{U_h} u^e(\mathbf{x})^2\mu(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{-c_1h}^{c_1h}\int_\Gamma [u^e(\mathbf{p}(\mathbf{x}), r)]^2 \ \mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x}))\mathrm{d}r \\ &= \int_{-c_1h}^{c_1h}\int_\Gamma [u(\mathbf{p}(\mathbf{x}), 0)]^2 \ \mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x}))\mathrm{d}r = 2c_1h\|u\|^2_{L^2(\Gamma)}.\end{aligned}$$

Combining this with (3.20) yields the result in (3.17).

From (3.8) we have that $u^e \in H^1(U_h)$. Note that

$$\int_{U_h} [\nabla u^e(\mathbf{x})]^2 \,\mu(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{-c_1 h}^{c_1 h}\int_\Gamma \left[(\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x}))\nabla_\Gamma u(\mathbf{p}(\mathbf{x}))\right]^2 \mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x}))\,\mathrm{d}r.$$

Using this in combination with $\|d(\mathbf{x})\mathbf{H}(\mathbf{x})\| \le \frac{1}{4}$ for all $\mathbf{x} \in U_h$ (cf. (3.13)) and the bounds in (3.20) we obtain the result in (3.18). Finally, using similar arguments and the bound in (3.10) one can derive the bound in (3.19). ∎

3.2. Approximation error bounds. Let $I_h : C(\overline{\omega_h}) \to V_h$ be the nodal interpolation operator. We use the approximation property of the linear finite element space V_h: For $v \in H^2(\omega_h)$

$$\|v - I_h v\|_{H^k(\omega_h)} \le C\,h^{2-k}\|v\|_{H^2(\omega_h)}, \quad k = 0, 1. \tag{3.21}$$

A consequence of this approximation result is given in the following lemma.

Lemma 3.3. *For $u \in H^2(\Gamma)$ and $k = 0, 1$ we have*

$$\|u^e - I_h u^e\|_{H^k(\omega_h)} \le C\,h^{\frac{5}{2}-k}\|u\|_{H^2(\Gamma)}. \tag{3.22}$$

Proof. From (3.21) and (3.18) we obtain

$$\|u^e - I_h u^e\|_{H^k(\omega_h)} \le C\,h^{2-k}\|u^e\|_{H^2(\omega_h)} \le C\,h^{2-k}\|u^e\|_{H^2(U_h)} \le C\,h^{\frac{5}{2}-k}\|u\|_{H^2(\Gamma)},$$

which proves the result. ∎

The following two lemmas play a crucial role in the analysis. In both lemmas we use a "pull back" strategy based on Lemma 3.1. For this we introduce a special local coordinate system as follows. For a subdomain $\omega \subset \mathbb{R}^n$ let $\rho(\omega)$ be the diameter of the largest ball that is contained in ω. Take an arbitrary planar segment T of Γ_h, i.e., $T \in \mathcal{F}_h$. Let $S_T \in \mathcal{T}_h$ be the tetrahedron such that $\Gamma_h \cap S_T = T$. There exists a planar extension T^e of T such that $T^e \subset U$, T^e is convex, $\mathbf{p}(S_T) \subset \mathbf{p}(T^e)$, and

$$\mathrm{diam}(T^e) \simeq \rho(T^e) \simeq h, \tag{3.23}$$

cf. Remark 8. This extension T^e is used to define a coordinate system in the neighborhood $N_T := \{\, \mathbf{x} \in U \mid \mathbf{p}(\mathbf{x}) \in \mathbf{p}(T^e)\,\}$. Note that $S_T \subset N_T$. Every $\mathbf{x} \in N_T$ has a unique decomposition of the form

$$\mathbf{x} = \mathbf{s} + \tilde{d}(\mathbf{x})\mathbf{n}(\mathbf{x}), \quad \text{with } \mathbf{s} \in T^e, \quad \tilde{d}(\mathbf{x}) := \pm\|\mathbf{s} - \mathbf{x}\|. \tag{3.24}$$

On which side of the plane T^e the point $\mathbf{x}$ lies determines the sign of $\tilde{d}(\mathbf{x})$. Note that $\tilde{d}$ is a signed distance, along the normal $\mathbf{n}(\mathbf{x})$, to the planar segment T^e. The representation in this coordinate system is denoted by Φ, i.e., $\Phi(\mathbf{x}) = (\mathbf{s}(\mathbf{x}), \tilde{d}(\mathbf{x}))$. This coordinate system is illustrated, for the 2D case, in Figure 3.1.

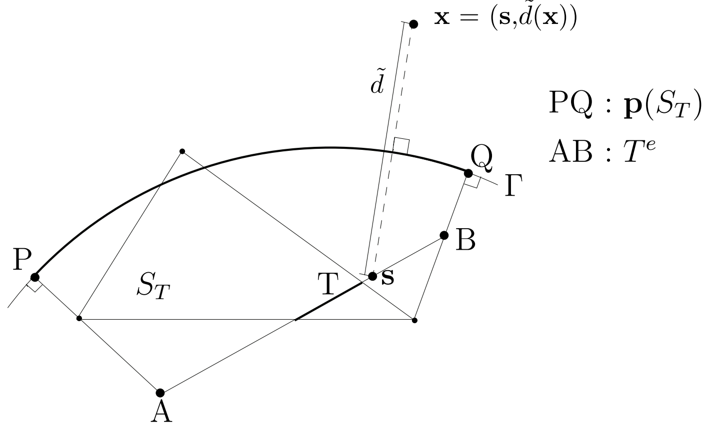

FIG. 3.1. *2D illustration of coordinate system.*

For $\mathbf{x} \in T^e$ we thus have $\Phi(\mathbf{x}) = (\mathbf{s}(\mathbf{x}), 0)$. Due to the shape-regularity of $\mathcal{T}_h$ there exists, in the Φ-coordinate system, a cylinder B_T that has the following properties:

$$B_T = T_b^e \times [d_0, d_1] \subset S_T, \quad T_b^e \subset T^e, \quad |T_b^e| \simeq h^2, \quad d_1 - d_0 \simeq h. \tag{3.25}$$

This coordinate system and the cylinder $B_T \subset S_T$ are used in the analysis below.

Remark 8. The following shows that an extension T^e of T with the properties described above exists. Take a fixed $\mathbf{x}_0 \in T$. Let W_Γ be the tangent plane at $\mathbf{p}(\mathbf{x}_0)$. The normal vector of W_Γ is $\mathbf{n}(\mathbf{x}_0)$. There is a subdomain w_Γ of this plane such that $\mathbf{p}(w_\Gamma) = \mathbf{p}(S_T)$. Due to the shape-regularity of $\mathcal{T}_h$ this subdomain is such that $\mathrm{diam}(w_\Gamma) \simeq \rho(w_\Gamma) \simeq h$ holds. Let $w_{\mathbf{x}_0}$ be a planar subdomain that is parallel to w_Γ, contains $\mathbf{x}_0$, and such that $\mathbf{p}(w_\Gamma) = \mathbf{p}(w_{\mathbf{x}_0})$. Using the assumption in (3.14) it follows that $\mathrm{diam}(w_{\mathbf{x}_0}) \simeq \rho(w_{\mathbf{x}_0}) \simeq h$ holds. The point $\mathbf{x}_0$ belongs to the planar subdomains $w_{\mathbf{x}_0}$ and T, which have normals $\mathbf{n}(\mathbf{x}_0)$ and $\mathbf{n}_h(\mathbf{x}_0)$, respectively. Due to assumption (3.15) the angle between these normals is bounded by ch, and thus there exists a planar extension $\tilde{T}^e$ of T such that $\tilde{T}^e \subset U$ and $\mathbf{p}(\tilde{T}^e) = \mathbf{p}(w_{\mathbf{x}_0})$, and now we set T^e to be a minimal convex envelope for $\tilde{T}^e$. This T^e has the property (3.23).

LEMMA 3.4. *Let v_h be a linear function on N_T and $u \in H^2(\Gamma)$. There exists a constant c independent of v_h, u, and T such that the following inequality holds:*

$$\|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(T^e)} \le ch^{-\frac{1}{2}}\|\nabla(u^e - v_h)\|_{L^2(S_T)} + h\|u\|_{H^2(\mathbf{p}(T^e))}. \tag{3.26}$$

Here ∇_{Γ_h} denotes the projection of the gradient on T^e.

Proof. Using Lemma 3.1, (3.6), and (3.10) we obtain

$$\begin{aligned} \|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(T^e)} &\le c\|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(T_b^e)} + ch^2\|\nabla^2_{\Gamma_h}u^e\|^2_{L^2(T^e)} \\ &\le c\|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(T_b^e)} + ch^2\|u\|^2_{H^2(\mathbf{p}(T^e))}. \end{aligned} \tag{3.27}$$

We consider the first term in (3.27). We write $\nabla v_h =: c_T$ and use the notation $\mathbf{x} = (\mathbf{s}(\mathbf{x}), \tilde{d}(\mathbf{x})) =: (\mathbf{s}, y)$ in the Φ-coordinate system. From (3.8) we have

$$\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) = \nabla u^e(\mathbf{s}, y) + d(\mathbf{x})\mathbf{H}(\mathbf{x})\nabla_\Gamma u(\mathbf{p}(\mathbf{x})).$$

Using this and (3.9) we obtain

$$\begin{aligned}
\|\nabla_{\Gamma_h}(u^e - v_h)\|^2_{L^2(T^e_b)} &= \|\nabla_{\Gamma_h} u^e - \mathbf{P}_h c_T\|^2_{L^2(T^e_b)} \\
&\leq 2\|\mathbf{P}_h(\nabla_\Gamma u)\circ\mathbf{p} - \mathbf{P}_h c_T\|^2_{L^2(T^e_b)} + 2\|\mathbf{P}_h d\mathbf{H}(\nabla_\Gamma u)\circ\mathbf{p}\|^2_{L^2(T^e_b)} \\
&\leq c\|(\nabla_\Gamma u)\circ\mathbf{p} - c_T\|^2_{L^2(T^e_b)} + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\
&= c\int_{T^e_b} \|\nabla_\Gamma u(\mathbf{p}(\mathbf{s},0)) - c_T\|^2\,\mathrm{d}\mathbf{s} + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\
&\leq ch^{-1}\int_{d_0}^{d_1}\int_{T^e_b} \|\nabla_\Gamma u(\mathbf{p}(\mathbf{s},0)) - c_T\|^2\,\mathrm{d}\mathbf{s}\mathrm{d}y + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\
&\leq ch^{-1}\int_{d_0}^{d_1}\int_{T^e_b} \|\nabla u^e(\mathbf{p}(\mathbf{s},y)) - c_T\|^2\,\mathrm{d}\mathbf{s}\mathrm{d}y + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\
&\leq ch^{-1}\|\nabla(u^e - v_h)\|^2_{L^2(B_T)} + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))} \\
&\leq ch^{-1}\|\nabla(u^e - v_h)\|^2_{L^2(S_T)} + ch^2\|u\|^2_{H^1(\mathbf{p}(S_T))}.
\end{aligned}$$

The combination of this result with the one in (3.27) completes the proof. ∎

LEMMA 3.5. *There are constants c_i independent of h such that for all $u \in H^2(\Gamma)$ and all $v_h \in V_h$ the following inequality holds:*

$$\|u^e - v_h\|_{L^2(\Gamma_h)} \leq c_1 h^{-\frac{1}{2}}\|u^e - v_h\|_{L^2(\omega_h)} + c_2 h^{\frac{1}{2}}\|u^e - v_h\|_{H^1(\omega_h)} + c_3 h^2\|u\|_{H^2(\Gamma)}. \tag{3.28}$$

Proof. We consider an arbitrary element $T \in \Gamma_h$. Let T^e be its extension as defined above. Take $v_h \in V_h$. The extension of v_h to a linear function on T^e is denoted by v_h, too. Using Lemma 3.1 and (3.6) we get:

$$\begin{aligned}
\|u^e - v_h\|^2_{L^2(T)} \leq \|u^e - v_h\|^2_{L^2(T^e)} &= \int_{T^e} (u^e(\mathbf{s},0) - v_h(\mathbf{s},0))^2\,\mathrm{d}\mathbf{s} \\
&\leq c\int_{T^e_b} (u^e(\mathbf{s},0) - v_h(\mathbf{s},0))^2\,\mathrm{d}\mathbf{s} \\
&\quad + ch^2\int_{T^e} \|\nabla_{\Gamma_h}\big(u^e(\mathbf{s},0) - v_h(\mathbf{s},0)\big)\|^2\,\mathrm{d}\mathbf{s}.
\end{aligned} \tag{3.29}$$

We consider the first term on the right-hand side of (3.29). For a linear function g and $0 \leq \delta_0 < \delta_1$ we have $g(\delta_i)^2 \leq \frac{6}{\delta_1-\delta_0}\int_{\delta_0}^{\delta_1} g(t)^2\,dt$ for $i = 0,1$ and $g(0) = g(\delta_0)\frac{\delta_1}{\delta_1-\delta_0} - g(\delta_1)\frac{\delta_0}{\delta_1-\delta_0}$. Hence, $|g(0)| \leq \frac{2\delta_1}{\delta_1-\delta_0}\max_{i=0,1}|g(\delta_i)|$ and thus

$$g(0)^2 \leq 24\left(\frac{\delta_1}{\delta_1-\delta_0}\right)^2 \frac{1}{\delta_1-\delta_0}\int_{\delta_0}^{\delta_1} g(t)^2\,dt \tag{3.30}$$

holds. Without loss of generality we can assume that d_0, d_1 from (3.25) satisfy $0 \leq d_0 < d_1$. Furthermore, we have $\frac{d_i}{d_1-d_0} \leq c$ for $i = 1,2$, with c independent of h. Using this and the result in (3.30) applied to the linear function $y \to c + v_h(\mathbf{s},y)$ we obtain

$$\begin{aligned}
&\int_{T^e_b} (u^e(\mathbf{s},0) - v_h(\mathbf{s},0))^2\,\mathrm{d}\mathbf{s} \leq ch^{-1}\int_{T^e_b}\int_{d_0}^{d_1} (u^e(\mathbf{s},0) - v_h(\mathbf{s},y))^2\,\mathrm{d}y\mathrm{d}\mathbf{s} \\
&= ch^{-1}\int_{T^e_b}\int_{d_0}^{d_1} (u^e(\mathbf{s},y) - v_h(\mathbf{s},y))^2\,\mathrm{d}y\,\mathrm{d}\mathbf{s} = ch^{-1}\|u^e - v_h\|^2_{L^2(B_T)} \\
&\leq ch^{-1}\|u^e - v_h\|^2_{L^2(S_T)}.
\end{aligned} \tag{3.31}$$

For the second term on the right-hand side of (3.29) we can apply Lemma 3.4, and thus we get

$$\|u^e - v_h\|_{L^2(T)}^2 \leq ch^{-1}\|u^e - v_h\|_{L^2(S_T)}^2 + ch\|\nabla(u^e - v_h)\|_{L^2(S_T)}^2 + ch^4\|u\|_{H^2(\mathbf{p}(T^e))}^2.$$

Summation over all triangles in $\mathcal{F}_h$ gives (3.28). ∎

LEMMA 3.6. *There are constants c_1, c_2 independent of h such that for all $u \in H^2(\Gamma)$ and all $v_h \in V_h$ the following inequality holds:*

$$\|u^e - v_h\|_{H^1(\Gamma_h)} \leq c_1 h^{-\frac{1}{2}}\|u^e - v_h\|_{H^1(\omega_h)} + c_2 h\|u\|_{H^2(\Gamma)}. \tag{3.32}$$

Proof. Take $u \in H^2(\Gamma)$ and $v_h \in V_h$. By definition of the H^1-norm on Γ_h we get

$$\|u^e - v_h\|_{H^1(\Gamma_h)}^2 = \|u^e - v_h\|_{L^2(\Gamma_h)}^2 + \|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(\Gamma_h)}^2.$$

For the first term on the right-hand side we can apply Lemma 3.5, and use

$$h^{-\frac{1}{2}}\|u^e - v_h\|_{L^2(\omega_h)} + c_2 h^{\frac{1}{2}}\|u^e - v_h\|_{H^1(\omega_h)} \leq ch^{-\frac{1}{2}}\|u^e - v_h\|_{H^1(\omega_h)}.$$

We now consider the second term

$$\|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(\Gamma_h)}^2 = \sum_{T \in \mathcal{F}_h} \|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(T)}^2.$$

Take a $T \in \mathcal{F}_h$ and extend v_h linearly outside T. This extension is denoted by v_h, too. Using Lemma 3.4 we get

$$\begin{aligned}\|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(T)}^2 &\leq \|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(T^e)}^2 \\ &\leq ch^{-1}\|\nabla(u^e - v_h)\|_{L^2(S_T)}^2 + h^2\|u\|_{H^2(\mathbf{p}(T^e))}^2.\end{aligned}$$

Summation over $T \in \mathcal{F}_h$ yields

$$\|\nabla_{\Gamma_h}(u^e - v_h)\|_{L^2(\Gamma_h)}^2 \leq c\, h^{-1}\|u^e - v_h\|_{H^1(\omega_h)}^2 + ch^2\|u\|_{H^2(\Gamma)}^2,$$

and thus the proof is completed. ∎

As a direct consequence of the previous two lemmas we obtain the following main theorem.

THEOREM 3.7. *For each $u \in H^2(\Gamma)$ the following hold:*

$$\inf_{v_h \in V_h^\Gamma} \|u^e - v_h\|_{L^2(\Gamma_h)} \leq \|u^e - (I_h u^e)_{|\Gamma_h}\|_{L^2(\Gamma_h)} \leq C\, h^2\|u\|_{H^2(\Gamma)}, \tag{3.33}$$

$$\inf_{v_h \in V_h^\Gamma} \|u^e - v_h\|_{H^1(\Gamma_h)} \leq \|u^e - (I_h u^e)_{|\Gamma_h}\|_{H^1(\Gamma_h)} \leq C\, h\|u\|_{H^2(\Gamma)}, \tag{3.34}$$

with a constant C independent of u and h.

Proof. Combine the results in the Lemmas 3.5 and 3.6 with the result in Lemma 3.3. ∎

3.3. Finite element error bounds. In this section we prove optimal discretization error bounds both in the $H^1(\Gamma_h)$- and the $L^2(\Gamma_h)$-norm. The arguments are very close to those in [5]. A difference is that in [5] the convergence results are derived in the $H^1(\Gamma)$- and the $L^2(\Gamma)$-norms by lifting the discrete solutions from Γ_h on Γ, whereas we consider the error between the finite element solution $u_h \in V_h^\Gamma$ and the extension u^e of the continuous solution to the discrete interface. This difference is of minor importance since error bounds in $H^1(\Gamma_h)$ imply similar bounds in $H^1(\Gamma)$, cf. Remark 9.

In the analysis we need a few results from [3]. For $\mathbf{x} \in \Gamma_h$ define $\tilde{\mathbf{P}}_h(\mathbf{x}) = \mathbf{I} - \mathbf{n}_h(\mathbf{x})\mathbf{n}(\mathbf{x})^T/(\mathbf{n}_h(\mathbf{x})^T\mathbf{n}(\mathbf{x}))$. In (2.19) in [3] the following representation of the surface gradient of $u \in H^1(\Gamma)$ in terms of $\nabla_{\Gamma_h} u^e$ is given:

$$\nabla_\Gamma u(\mathbf{p}(\mathbf{x})) = \big(\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x})\big)^{-1}\tilde{\mathbf{P}}_h(\mathbf{x})\nabla_{\Gamma_h} u^e(\mathbf{x}) \quad \text{a.e. on } \Gamma_h. \tag{3.35}$$

For $\mathbf{x} \in \Gamma_h$ define

$$\mu_h(\mathbf{x}) = (1 - d(\mathbf{x})\kappa_1(\mathbf{x}))(1 - d(\mathbf{x})\kappa_1(\mathbf{x}))\mathbf{n}(\mathbf{x})^T\mathbf{n}_h(\mathbf{x}).$$

The integral transformation formula

$$\mu_h(\mathbf{x})\mathrm{d}\mathbf{s}_h(\mathbf{x}) = \mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x})), \quad \mathbf{x} \in \Gamma_h \tag{3.36}$$

holds, where $\mathrm{d}\mathbf{s}_h(\mathbf{x})$ and $\mathrm{d}\mathbf{s}(\mathbf{p}(\mathbf{x}))$ are the surface measures on Γ_h and Γ, respectively, cf. (2.20) in [3]. From

$$\|\mathbf{n}(\mathbf{x}) - \mathbf{n}_h(\mathbf{x})\|^2 = 2\big(1 - \mathbf{n}(\mathbf{x})^T\mathbf{n}_h(\mathbf{x})\big),$$

the assumption in (3.15) and $|d(\mathbf{x})| \le ch^2,\ |\kappa_i(\mathbf{x})| \le c$ we obtain

$$\text{ess sup}_{\mathbf{x}\in\Gamma_h}|1 - \mu_h(\mathbf{x})| \le ch^2, \tag{3.37}$$

with a constant c independent of h.

THEOREM 3.8. *Let $u \in H^2(\Gamma)$ be the solution of (2.2), and $u_h \in V_h^\Gamma$ the solution of (2.8), with $f_h = f^e - c_f$, where c_f is such that $\int_{\Gamma_h} f_h \,\mathrm{d}\mathbf{s} = 0$. The following discretization error bound holds:*

$$\|\nabla_{\Gamma_h}(u^e - u_h)\|_{L^2(\Gamma_h)} \le c\,h\,\|f\|_{L^2(\Gamma)}, \tag{3.38}$$

with a constant c independent of f and h.

Proof. Using (3.37) we obtain $\left|1 - \frac{1}{\mu_h(\mathbf{x})}\right| \le ch^2$ on Γ_h. Define

$$c_f := \int_{\Gamma_h} f^e \,\mathrm{d}\mathbf{s}_h, \quad \delta_f := (1 - \mu_h)f^e - c_f.$$

Note that $f_h = f^e - c_f$ and due to $\int_\Gamma f\,\mathrm{d}\mathbf{s} = 0$ we get

$$|c_f| = \left|\int_{\Gamma_h} f^e \,\mathrm{d}\mathbf{s}_h\right| = \left|\int_\Gamma f(\frac{1}{\mu_h} - 1)\,\mathrm{d}\mathbf{s}\right| \le ch^2\|f\|_{L^2(\Gamma)}.$$

Furthermore,

$$\|\delta_f\|_{L^2(\Gamma_h)} \le \text{ess}\sup_{\mathbf{x}\in\Gamma_h}|1 - \mu_h(\mathbf{x})|\|f^e\|_{L^2(\Gamma_h)} + |\Gamma_h|^{\frac{1}{2}}|c_f| \le ch^2\|f\|_{L^2(\Gamma)}. \tag{3.39}$$

Using relation (3.35) and (3.36) we obtain

$$\int_\Gamma \nabla_\Gamma u \nabla_\Gamma v \,\mathrm{d}\mathbf{s} = \int_{\Gamma_h} \mathbf{A}_h \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} v^e \,\mathrm{d}\mathbf{s}_h \quad \text{for all } v \in H^1(\Gamma), \tag{3.40}$$

with $\mathbf{A}_h(\mathbf{x}) = \mu_h(\mathbf{x})\tilde{\mathbf{P}}_h(\mathbf{x})(\mathbf{I} - d(\mathbf{x})\mathbf{H}(\mathbf{x}))^{-2}\tilde{\mathbf{P}}_h(\mathbf{x})$. Any $\psi_h \in H^1(\Gamma_h)$ can be lifted on Γ by defining $\psi_h^l(\mathbf{p}(\mathbf{x})) := \psi_h(\mathbf{x})$. Then $\psi_h^l \in H^1(\Gamma)$ holds. From the definition of the discrete solution u_h in (2.8) we get, for arbitrary $\psi_h \in V_h^\Gamma$,

$$\begin{aligned}\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \nabla_{\Gamma_h} \psi_h \, \mathrm{d}\mathbf{s}_h &= \int_{\Gamma_h} f_h \psi_h \, \mathrm{d}\mathbf{s}_h = \int_{\Gamma} (f - c_f)\mu_h(\mathbf{x})^{-1} \psi_h^l \, \mathrm{d}\mathbf{s} \\ &= \int_{\Gamma} f \psi_h^l \, \mathrm{d}\mathbf{s} + \int_{\Gamma_h} \delta_f \psi_h \, \mathrm{d}\mathbf{s}_h \\ &= \int_{\Gamma} \nabla_\Gamma u \nabla_\Gamma \psi_h^l \, \mathrm{d}\mathbf{s} + \int_{\Gamma_h} \delta_f \psi_h \, \mathrm{d}\mathbf{s}_h \\ &= \int_{\Gamma_h} \mathbf{A}_h \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} \psi_h \, \mathrm{d}\mathbf{s}_h + \int_{\Gamma_h} \delta_f \psi_h \, \mathrm{d}\mathbf{s}_h.\end{aligned}$$

Using this we obtain, for arbitrary $\psi_h \in V_h^\Gamma$,

(3.41)
$$\begin{aligned}\int_{\Gamma_h} \nabla_{\Gamma_h}(u^e - u_h) \nabla_{\Gamma_h} \psi_h \, \mathrm{d}\mathbf{s}_h &= \int_{\Gamma_h} (\mathbf{I} - \mathbf{A}_h) \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} \psi_h \, \mathrm{d}\mathbf{s}_h - \int_{\Gamma_h} \delta_f \psi_h \, \mathrm{d}\mathbf{s}_h \\ &= \int_{\Gamma_h} \mathbf{P}_h(\mathbf{I} - \mathbf{A}_h) \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h} \psi_h \, \mathrm{d}\mathbf{s}_h - \int_{\Gamma_h} \delta_f \psi_h \, \mathrm{d}\mathbf{s}_h.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\|\nabla_{\Gamma_h}(u^e - u_h)\|_{L^2(\Gamma_h)}^2 &= \int_{\Gamma_h} \nabla_{\Gamma_h}(u^e - u_h) \nabla_{\Gamma_h}(u^e - \psi_h) \; ds_h \\ &\quad + \int_{\Gamma_h} \mathbf{P}_h(\mathbf{I} - \mathbf{A}_h) \nabla_{\Gamma_h} u^e \nabla_{\Gamma_h}(\psi_h - u_h) \; ds_h \\ &\quad - \int_{\Gamma_h} \delta_f(\psi_h - u_h) \; ds_h.\end{aligned} \tag{3.42}$$

From $\|\tilde{\mathbf{P}}_h - \mathbf{A}_h\| \le ch^2$ a.e. on Γ_h and $\mathbf{P}_h \tilde{\mathbf{P}}_h = \mathbf{P}_h$ we obtain, for $x \in \Gamma_h$,

$$\|\mathbf{P}_h(x)(\mathbf{I} - \mathbf{A}_h(x))\| = \|\mathbf{P}_h(x)(\tilde{\mathbf{P}}_h(x) - \mathbf{A}_h(x))\| \le ch^2. \tag{3.43}$$

Furthermore, using (3.9) we get

$$\begin{aligned}\|\nabla_{\Gamma_h} u^e\|_{L^2(\Gamma_h)} &\le \operatorname{ess}\sup_{x \in \Gamma_h} \|\mathbf{P}_h(x)(\mathbf{I} - d\mathbf{H}(x))\| \|\nabla_\Gamma u\|_{L^2(\Gamma)} \\ &\le c\|f\|_{L^2(\Gamma)}.\end{aligned} \tag{3.44}$$

We introduce the notation $E_h := \|\nabla_{\Gamma_h}(u^e - u_h)\|_{L^2(\Gamma_h)}$. Note that by taking $\psi_h = (I_h u^e)_{|\Gamma_h}$ and using the approximation result (3.34) we have

$$\|\nabla_{\Gamma_h}(u_h - \psi_h)\|_{L^2(\Gamma_h)} \le E_h + \|\nabla_{\Gamma_h}(u^e - \psi_h)\|_{L^2(\Gamma_h)} \le E_h + ch\|f\|_{L^2(\Gamma)}.$$

For the third term on the right-hand side in (3.42) we have the bound

$$\begin{aligned}\left|\int_{\Gamma_h} \delta_f(\psi_h - u_h) \; ds_h\right| &\le \|\delta_f\|_{L^2(\Gamma_h)} \|\psi_h - u_h\|_{L^2(\Gamma_h)} \\ &\le ch^2 \|f\|_{L^2(\Gamma)} \big(\|\psi_h - u^e\|_{L^2(\Gamma_h)} + \|u^e - u_h\|_{L^2(\Gamma_h)}\big) \\ &\le ch^2 \|f\|_{L^2(\Gamma)} \big(ch^2 \|f\|_{L^2(\Gamma)} + \|u^e - u_h\|_{L^2(\Gamma_h)}\big).\end{aligned}$$

Note that

$$\begin{aligned}\|u^e - u_h\|_{L^2(\Gamma_h)} &\leq \|u^e\|_{L^2(\Gamma_h)} + \|u_h\|_{L^2(\Gamma_h)} \leq c\big(\|u\|_{L^2(\Gamma)} + \|\nabla_{\Gamma_h} u_h\|_{L^2(\Gamma_h)}\big)\\ &\leq c\big(\|\nabla_\Gamma u\|_{L^2(\Gamma)} + \|\nabla_{\Gamma_h} u_h\|_{L^2(\Gamma_h)}\big) \leq c\|f\|_{L^2(\Gamma)}.\end{aligned}$$

The combination of these results leads to

$$\begin{aligned}E_h^2 &\leq E_h ch\|f\|_{L^2(\Gamma)} + ch^2\|f\|_{L^2(\Gamma)}\big(E_h + ch\|f\|_{L^2(\Gamma)} + \|f\|_{L^2(\Gamma)}\big)\\ &\leq \frac{1}{2}E_h^2 + ch^2\|f\|^2_{L^2(\Gamma)}.\end{aligned}$$

This yields the bound in (3.38). □

Remark 9. We indicate how the error bound (3.38) in $H^1(\Gamma_h)$ yields a similar bound in $H^1(\Gamma)$. For this we need the extension of functions defined on Γ_h along the normals $\mathbf{n}$ on Γ: For $v \in C(\Gamma_h)$ we define, for $\mathbf{x} \in \Gamma_h$,

$$v^{e,h}(\mathbf{x} + \alpha\mathbf{n}(\mathbf{x})) := v(\mathbf{x}) \quad \text{for all} \quad \alpha \in \mathbb{R}, \quad \text{with} \quad \mathbf{x} + \alpha\mathbf{n}(\mathbf{x}) \in U.$$

The following holds (cf. [3], Lemma 3.3 in [8]):

$$\|\nabla_\Gamma v^{e,h}\|_{L^2(\Gamma)} \leq c\|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \quad \text{for all} \quad v \in H^1(\Gamma_h) \cap C(\Gamma_h).$$

Using this for the error $v = u^e - u_h$ and noting that $(u^e)^{e,h} = u$ on Γ the bound (3.38) yields

$$\|\nabla_\Gamma(u - u_h^{e,h})\|_{L^2(\Gamma)} \leq ch\|f\|_{L^2(\Gamma)},$$

i.e., an optimal error bound in $H^1(\Gamma)$.

We now apply a duality argument to obtain an $L^2(\Gamma_h)$ error bound.

THEOREM 3.9. *Let u and u_h be as in Theorem* 3.8. *The following error bound holds*

$$\|u^e - u_h\|_{L^2(\Gamma_h)} \leq c\, h^2\, \|f\|_{L^2(\Gamma)}, \tag{3.45}$$

with a constant c independent of f and h.

Proof. Denote $e_h := (u^e - u_h)|_{\Gamma_h}$ and let e_h^l be the lift of e_h on Γ and $c_e := \int_\Gamma e_h^l \,\mathrm{d}\mathbf{s}$. Consider the problem: Find $w \in H^1(\Gamma)$, with $\int_\Gamma w \,\mathrm{d}\mathbf{s} = 0$, such that

$$\int_\Gamma \nabla_\Gamma w \nabla_\Gamma v \,\mathrm{d}\sigma = \int_\Gamma (e_h^l - c_e) v \,\mathrm{d}\mathbf{s} \qquad \text{for all} \quad v \in H^1(\Gamma). \tag{3.46}$$

The solution w satisfies $w \in H^2(\Gamma)$ and $\|w\|_{H^2(\Gamma)} \leq c\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$, with $\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}} := \|e_h^l - c_e\|_{L^2(\Gamma)}$. Furthermore, $\|\nabla_{\Gamma_h} w^e\|_{L^2(\Gamma_h)} \leq c\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$ and $\|w^e\|_{L^2(\Gamma_h)} \leq c\|w\|_{L^2(\Gamma)} \leq c\|\nabla_\Gamma w\|_{L^2(\Gamma)} \leq c\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$. Due to (3.46) and (3.41) we have, for any $\psi_h \in V_h^\Gamma$,

$$\begin{aligned}\|e_h^l\|^2_{L^2(\Gamma)/\mathbb{R}} &= \int_\Gamma \nabla_\Gamma w \nabla_\Gamma (e_h^l - c_e)\,\mathrm{d}\mathbf{s} = \int_\Gamma \nabla_\Gamma w \nabla_\Gamma e_h^l \,\mathrm{d}\mathbf{s} = \int_{\Gamma_h} \mathbf{A}_h \nabla_{\Gamma_h} e_h \nabla_{\Gamma_h} w^e \,\mathrm{d}\mathbf{s}_h\\ &= \int_{\Gamma_h} \nabla_{\Gamma_h} e_h \nabla_{\Gamma_h}(w^e - \psi_h)\,\mathrm{d}\mathbf{s}_h + \int_{\Gamma_h} \mathbf{P}_h(\mathbf{A}_h - \mathbf{I})\nabla_{\Gamma_h} e_h \nabla_{\Gamma_h} w^e \,\mathrm{d}\mathbf{s}_h\\ &\quad + \int_{\Gamma_h} \mathbf{P}_h(\mathbf{I} - \mathbf{A}_h)\nabla_{\Gamma_h} u^e \nabla_{\Gamma_h}\psi_h \,\mathrm{d}\mathbf{s}_h - \int_{\Gamma_h} \delta_f \psi_h \,\mathrm{d}\mathbf{s}_h.\end{aligned}$$

We introduce $E_h := \|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$. Thanks to the approximation property (3.34) one can choose ψ_h such that $\|\nabla_{\Gamma_h}(w^e - \psi_h)\|_{L^2(\Gamma_h)} \leq ch\|w\|_{H^2(\Gamma)} \leq chE_h$. Using Cauchy–Schwarz and triangle inequalities and the bounds in (3.39), (3.43) we get

$$\begin{aligned} E_h^2 &\leq \|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)} chE_h + ch^2\|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)}\|\nabla_{\Gamma_h} w^e\|_{L^2(\Gamma_h)} \\ &\quad + ch^2\|\nabla_{\Gamma_h} u^e\|_{L^2(\Gamma_h)}\big(\|\nabla_{\Gamma_h} w^e\|_{L^2(\Gamma_h)} + chE_h\big) + ch^2\|f\|_{L^2(\Gamma)}\big(\|w^e\|_{L^2(\Gamma_h)} + chE_h\big) \\ &\leq ch^2\|f\|_{L^2(\Gamma)}E_h + ch^2\|f\|_{L^2(\Gamma)}\big(E_h + chE_h\big). \end{aligned}$$

Hence, $E_h \leq ch^2\|f\|_{L^2(\Gamma)}$ holds. We have

$$|c_e| = \left|\int_\Gamma u - u_h^e \, d\mathbf{s}\right| = \left|\int_\Gamma u_h^e \, d\mathbf{s}\right| = \left|\int_{\Gamma_h} (\mu_h - 1)u_h^e \, d\mathbf{s}_h\right| \leq ch^2\|f\|_{L^2(\Gamma)},$$

and thus

$$\|e_h\|_{L^2(\Gamma_h)} \leq c\left\|\mu_h^{-\frac{1}{2}} e_h\right\|_{L^2(\Gamma_h)}^2 = c\|e_h^l\|_{L^2(\Gamma)} \leq c(E_h + |c_e|) \leq ch^2\|f\|_{L^2(\Gamma)},$$

which completes the proof. ∎

4. Numerical experiments. In this section we present results of numerical experiments. As a first test problem we consider the Laplace–Beltrami equation on the unit sphere:

$$-\Delta_\Gamma u = f \quad \text{on } \Gamma,$$

with $\Gamma = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\|_2 = 1\}$ and $\Omega = (-2, 2)^3$.

The source term f is taken such that the solution is given by

$$u(\mathbf{x}) = \frac{a}{\|\mathbf{x}\|^3}\left(3x_1^2 x_2 - x_2^3\right), \quad \mathbf{x} = (x_1, x_2, x_3) \in \Omega,$$

with $a = 12$. Using the representation of u in spherical coordinates one can verify that u is an eigenfunction of $-\Delta_\Gamma$:

$$u(r, \phi, \theta) = a\sin(3\phi)\sin^3\theta, \qquad -\Delta_\Gamma u = 12u =: f(r, \phi, \theta). \tag{4.1}$$

The right-hand side f satisfies the compatibility condition $\int_\Gamma f \, d\mathbf{s} = 0$, likewise does u. Note that u and f are constant along normals at Γ.

A family $\{\mathcal{T}_l\}_{l\geq 0}$ of tetrahedral triangulations of Ω is constructed as follows. We triangulate Ω by starting with a uniform subdivision into 48 tetrahedra with mesh size $h_0 = \sqrt{3}$. Then we apply an adaptive red-green refinement algorithm (implemented in the software package DROPS [4]) in which in each refinement step the tetrahedra that contain Γ are refined such that on level $l = 1, 2, \ldots,$ we have

$$h_T \leq \sqrt{3}\, 2^{-l} \quad \text{for all } T \in \mathcal{T}_l, \quad \text{with } \ T \cap \Gamma \neq \emptyset.$$

The family $\{\mathcal{T}_l\}_{l\geq 0}$ is consistent and shape-regular. The interface Γ is the zero-level of $\varphi(\mathbf{x}) := \|\mathbf{x}\|^2 - 1$. Let $\varphi_h := I(\varphi)$, where I is the standard nodal interpolation operator on $\mathcal{T}_l$. The discrete interface is given by $\Gamma_{h_l} := \{\, \mathbf{x} \in \Omega \mid I(\phi_h)(\mathbf{x}) = 0 \,\}$, cf. (2.4). Let $\{\phi_i\}_{1\leq i\leq m}$ be the nodal basis functions corresponding to the vertices of the tetrahedra in ω_h, as explained in Remark 2. The entries $\int_{\Gamma_h} \nabla_{\Gamma_h}\phi_i \cdot \nabla_{\Gamma_h}\phi_j d\mathbf{s}_h$ of the stiffness matrix are computed within machine accuracy. For the right-hand side of the

Galerkin discretization (2.8) we need an extension f_h of f. In order to be consistent with the theoretical analysis we take the constant extension of f along the normals at Γ, i.e., we take $f_h(r,\phi,\theta) = f(1,\phi,\theta) + c_h$, with $f(r,\phi,\theta)$ as in (4.1) and c_h such that $\int_{\Gamma_h} f_h \, \mathrm{d}\mathbf{s}_h = 0$. For the computation of the integrals $\int_T f_h \psi_h \, \mathrm{d}\mathbf{s}_h$ we use a quadrature-rule that is exact up to order five. The computed solution u_h is normalized such that $\int_{\Gamma_h} u_h \, \mathrm{d}\mathbf{s}_h = 0$.

The discrete problem is solved using a standard CG method with a symmetric Gauss–Seidel preconditioner to a relative tolerance of 10^{-6}. The number of iterations needed on level $l = 1, 2, \ldots, 7$, is 14, 25, 50, 101, 209, 417, 837, respectively.

The discretization errors in the $L^2(\Gamma_h)$ norm are given in Table 4.1. The extension u^e of u is given by $u^e(r,\phi,\theta) := u(1,\phi,\theta)$, cf. (4.1).

TABLE 4.1
Discretization errors and error reduction.

level l	$\|u^e - u_h\|_{L^2(\Gamma_h)}$	factor
1	0.4418	–
2	0.1149	3.85
3	0.02965	3.88
4	0.007298	4.06
5	0.001865	3.91
6	0.0004629	4.03
7	0.0001158	4.00

These results clearly show the h^2 behavior as predicted by our theoretical analysis. To illustrate the fact that in this approach the triangulation of the approximate manifold Γ_h is strongly shape-irregular we show a part of this triangulation in Figure 4.1.

The discrete solution is visualized in Figure 4.2.

To demonstrate the flexibility of the method with respect to the form of Γ we repeat the previous experiment but now with a torus instead of the unit sphere. $\Gamma \subset \Omega = (-2,2)^3$, with $\Gamma = \{\mathbf{x} \in \Omega \mid r^2 = x_3^2 + (\sqrt{x_1^2 + x_2^2} - R)^2\}$. We take $R = 1$ and $r = 0.6$. In the coordinate system (ρ, ϕ, θ), with

$$\mathbf{x} = R \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} + \rho \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ \sin\theta \end{pmatrix},$$

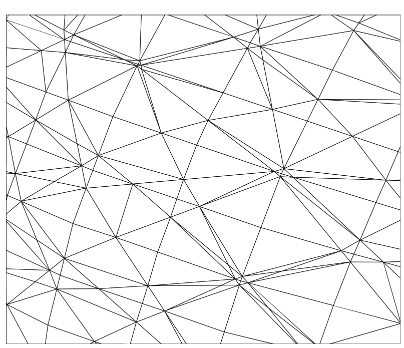

FIG. 4.1. *Details of the induced triangulation of* Γ_h.

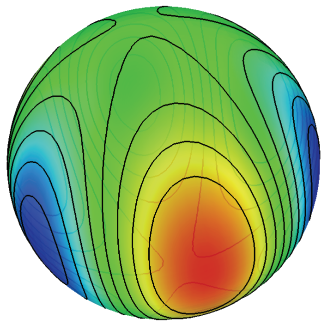

FIG. 4.2. *Level lines of the discrete solution u_h.*

the ρ-direction is normal to Γ, $\frac{\partial \mathbf{x}}{\partial \rho} \perp \Gamma$ for $\mathbf{x} \in \Gamma$. Thus, the following solution u and corresponding right-hand side f are constant in the normal direction:

$$
\begin{aligned}
u(\mathbf{x}) &= \sin(3\phi)\cos(3\theta+\phi), \\
f(\mathbf{x}) &= r^{-2}(9\sin(3\phi)\cos(3\theta+\phi)) \\
&\quad - (R + r\cos(\theta))^{-2}(-10\sin(3\phi)\cos(3\theta+\phi) - 6\cos(3\phi)\sin(3\theta+\phi)) \\
&\quad - (r(R + r\cos(\theta))^{-1}(3\sin(\theta)\sin(3\phi)\sin(3\theta+\phi)).
\end{aligned} \tag{4.2}
$$

Both u and f satisfy the zero mean compatibility condition.

The discretization errors in the $L^2(\Gamma_h)$-norm are given in Table 4.2. The extension u^e of u is given by $u^e(\rho,\phi,\theta) := u(r,\phi,\theta)$, cf. (4.2). Again, we observe the h^2 behavior as predicted by the theoretical analysis. The discrete solution is visualized in Figure 4.3.

TABLE 4.2
Torus: Discretization errors and error reduction.

level l	$\|u^e - u_h\|_{L^2(\Gamma_h)}$	factor
1	1.699	–
2	0.5292	3.21
3	0.1402	3.77
4	0.03632	3.86
5	0.009317	3.90
6	0.002298	4.05
7	0.0005711	4.02

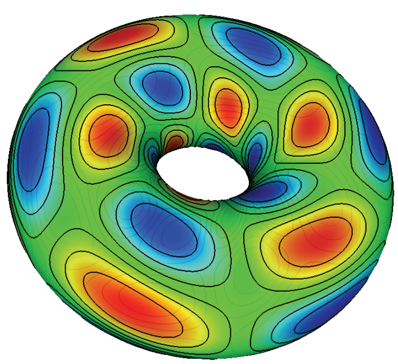

FIG. 4.3. *Torus: Level lines of the discrete solution u_h.*

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