NON-SEPARABLE RADIAL FRAME MULTIRESOLUTION ANALYSIS IN MULTIDIMENSIONS

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ABSTRACT. In this paper we present a non separable multiresolution structure based on frames which is defined by radial frame scaling functions. The Fourier transform of these functions is the indicator (characteristic) function of a measurable set. We also construct the resulting frame multiwavelets, which can be isotropic as well. Our construction can be carried out in any number of dimensions and for a big variety of dilation matrices.

1. INTRODUCTION AND PRELIMINARIES

Let H be a complex Hilbert space. A *unitary system* $\mathcal{U}$ is a set of unitary operators acting on H which contains the identity operator I on H. Now, let D be the *(dyadic) Dilation operator*

$$(Df)(\mathbf{t}) = 2^{n/2} f(2\mathbf{t}), \quad f \in L^2(\mathbb{R}^n) \tag{1}$$

and $T_{\mathbf{k}}$ be the Translation operator defined by

$$(T_{\mathbf{k}} f)(\mathbf{t}) = f(\mathbf{t} - \mathbf{k}), \quad f \in L^2(\mathbb{R}^n), \ \ \mathbf{k} \in \mathbb{Z}^n. \tag{2}$$

We refer to the unitary system $\mathcal{U}_{D,\mathbb{Z}^n} := \{D^j T_{\mathbf{k}} : j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$ as the *n-dimensional separable Affine system*. This system has been extensively used in wavelet analysis for the construction of separable wavelet bases. In fact only a few non-separable wavelet bases have been constructed and all these examples were exclusively given in two dimensions. However, an important drawback of these families of wavelets is the absence of enough symmetry, differentiability and the absolute lack of isotropy. These examples were also given with respect to a small class of dilation operators and all of them are compactly supported in the time domain. Apparently the whole issue of designing wavelet bases in multidimensions still remains a mostly unexplored area, full of challenges and revealing interesting and surprising results.

The motivation for the present paper stems from the following elementary observation: The Fourier transform of the Shannon scaling function (the scaling

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function of the Shannon MRA) is the indicator (characteristic) function of the interval $[-1/2, 1/2)$. Apperently, this function is even. Keeping in mind that even functions are also radial (a function is radial if it depends only on the radial variable) one might wonder, what is the multidimensional analogue of even, Shannon-like scaling functions (scaling functions whose Fourier transform is the characteristic function of measuarable subset of $\mathbb{R}^n$). This particular problem motivated us to introduce the radial frame multiresolution analysis. Our construction is based on a very general multiresolution scheme of abstract Hilbert spaces developed by Papadakis in [18], namely the Generalized Frame Multiresolution Analysis (GFMRA). The main characteristic of GFMRAs is that they can be generated by redundant sets of frame scaling functions. In fact, GFMRAs encompass all classical MRAs in one and multidimensions as well as the FMRAs of Benedetto and Li ([6]).

In this paper we construct non-separable Shannon-like GFMRAs of $L^2(\mathbb{R}^n)$ whose scaling functions are radial and are defined with respect to certain unitary systems, which we will later introduce. We also derive certain of their associated frame multiwavelet sets. Scaling functions for $L^2(\mathbb{R}^n)$ with $n \geq 2$ that are radial have not been constructed in the past. However, certain classes of non separable scaling functions in two dimensions, with some continuity properties with respect to dyadic dilations or dilations induced by the Quincunx matrix only have been constructed in the past (e.g. [8, 14, 13, 10], [4]). All of them have no axial symmetries and are not smooth, except those constructed in [5], which can be made arbitrarily smooth, but are highly asymmetric. Another construction in the spirit of digital filter design, but not directly related to wavelets can be found in [1] and [20]. These two and the ridgelets and beamlets ([7, 9, 21]) share two properties of our Radial GFMRAs: the separability of the designed filters with respect to polar coordinates and the redundancy of the induced representations. However, our construction in contrast to those due to Simoncelli et. al., Candes, Donoho, Starck et al. are in the spirit of classical multiresolution analysis and can be extended to any number of dimensions and with respect to a great variety of dilation matrices.

The merit of non separable wavelets and scaling functions is that the resulting processing of images is more compatible with that of human or mammallian vision, because mammals do not process images vertically and horizontally as separable filter banks resulting from separable multiresolution analyses do ([22]). As Marr suggests in his book [15] our visual system critically depends on edge detection. In order to model this detection Marr and Hildreth used the Laplacian operator which is the "lowest order isotropic operator" ([16]), because our visual system is orientation insensitive to edge detection. This suggests that perhaps the most desirable property in filter design for image processing is the isotropy of the filter. Thus radial scaling functions for multiresolutions based on frames are the best (and, according to proposition 5 the only) types of image processing filters that meet the isotropy requirement.

Before we proceed we need a few definitions and results from [18].

The family $\{x_i : i \in I\}$ is a *frame* for the Hilbert space H if there exist constants $A, B > 0$ such that for every $x \in H$ we have

$$A \|x\|^2 \leq \sum_{i \in I} |\langle x, x_i \rangle|^2 \leq B \|x\|^2 \ .$$

We refer to the positive constants A, B as *frame bounds*. Apparently for every frame its bounds are not uniquely defined. We refer to the frame as *tight* if $A = B$ and as *Parseval frame* if $A = B = 1$. A frame $\{x_i : i \in I\}$ of H is called *exact* if each one of its proper subsets is not a frame for H. Riesz bases are exact frames and vice-versa. The operator S defined by

$$Sx = \{\langle x, x_i \rangle\}_{i \in I} \quad x \in H$$

is called the *analysis operator* corresponding to the frame $\{x_i : i \in I\}$. Using this operator we can construct the dual frame $\{x_i' : i \in I\}$ of $\{x_i : i \in I\}$ by setting $x_i' := (S^*S)^{-1} x_i$. Then, for every $x \in H$ we have

$$x = \sum_i \langle x, x_i' \rangle x_i.$$

We are interested in unitary systems $\mathcal{U}$ of the form $\mathcal{U} = \mathcal{U}_0 G$, where $\mathcal{U}_0 = \{U^j : j \in \mathbf{Z}\}$ and G is an abelian unitary group. We will often refer to G as a *translation group*. Obviously unitary systems of this form generalize the affine system.

Definition 1. *A sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of an abstract Hilbert space H is a Generalized Frame Multiresolution Analysis of H if it is increasing, i.e. $V_j \subseteq V_{j+1}$ for every $j \in \mathbb{Z}$ and satisfies the following properties:*

(a) $V_j = U^j(V_0)$, $j \in \mathbb{Z}$

(b) $\bigcap_j V_j = \{0\}$, $\overline{\bigcup_j V_j} = H$

(c) There exists a countable subset B of V_0 such that the set $G(B) = \{g\phi : g \in G, \phi \in B\}$ is a frame of V_0.

Every such set B is called a frame multiscaling set for $\{V_j\}_j$. Every subset C of V_1 such that $G(C) = \{g\psi : g \in G, \psi \in C\}$ is a frame of $W_0 := V_1 \cap V_0^{\perp}$ is called a semiorthogonal frame multiwavelet vector set associated with $\{V_j\}_j$.

As it was observed in [18] $G(B')$ is the canonical dual of $G(B)$, where $B' := \{(S^*S)^{-1}\phi : \phi \in B\}$, where S is the analysis operator corresponding to the frame $G(B)$. Likewise the canonical dual of $G(C)$ is the family $G(C')$, where $C' := \{(S^*S)^{-1}\psi : \psi \in C\}$ and S is again the analysis operator corresponding to the frame $G(B)$. We refer to B' and as the *dual frame scaling set* corresponding to B and to C' as the *dual frame wavelet set* corresponding to C.

If B is a singleton we refer to its unique element as a *frame scaling vector* and, if $H = L^2(\mathbb{R}^n)$, as a *frame scaling function*. We also let $W_j := U^j(W_0)$, for every $j \in \mathbb{Z}$. Note, that if C is a semiorthogonal frame multiwavelet vector set associated with the GFMRA $\{V_j\}_j$ then the set $\{D^j g\psi : j \in \mathbb{Z}, g \in G, \psi \in C\}$ is a frame for H with the same frame bounds as the frame $G(C)$.

In order to accomplish the construction of the frame multiwavelet sets associated with a GFMRA $\{V_j\}_j$ we need the following additional hypotheses.

- There exists a mapping $\sigma : G \to G$ satisfying
$$gD = D\sigma(g) \ , \quad \text{for every } g \in G.$$
This particular assumption implies that σ is an injective homomorphism and $\sigma(G)$ is a subgroup of G. (See [11] for proofs)
- $|G : \sigma(G)| = n < +\infty$, where $|G : \sigma(G)|$ is the index of the subgroup $\sigma(G)$.

As mentioned before for the purposes of our study we will exclusively use multidimensional affine unitary systems. Before proceeding we need the following definition:

Definition 2. *An $n \times n$ invertible matrix A is expanding if all its entries are real and all its eigenvalues have modulus greater than* 1. *A Dilation matrix is an expanding matrix that leaves $\mathbb{Z}^n$ invariant, i.e. $A(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$.*

The previous definition readily yields the following observations:

- All the entries of a dilation matrix are integers, because such a matrix leaves $\mathbb{Z}^n$ invariant.
- The previous observation implies that $detA$ is an integer.

The multidimensional affine unitary systems we are interested in are the systems of the form $\mathcal{U}_0 G$, where $\mathcal{U}_0$ is the cyclic torsion free group generated by a dilation operator D defined by
$$Df(\mathbf{t}) = |detA|^{1/2} f(A\mathbf{t}), \quad f \in L^2(\mathbb{R}^n)$$
where A is a dilation matrix and $G = \{T_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^n\}$. Obviously, G is isomorphic with $\mathbb{Z}^n$. Using the definitions of translations and dilations one can easily verify $T_{\mathbf{k}} D = D T_{A\mathbf{k}}$, thus $\sigma(T_{\mathbf{k}}) = T_{A\mathbf{k}}$, for every $\mathbf{k} \in \mathbb{Z}^n$. Note that σ is legitimally defined, because $A(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$. Apparently, the quotient group $G/\sigma(G)$ is homeomorphically isomorphic with $\mathbb{Z}^n/A(\mathbb{Z}^n)$. Thus we have $|G : \sigma(G)| = |\mathbb{Z}^n : A(\mathbb{Z}^n)| = |detA|$. Now, set $\mathbf{q}_0 = \mathbf{0}$, $p := |detA|$ and fix $\mathbf{q}_r \in \mathbb{Z}^n$, for $r = 1, 2, \ldots, p-1$ so that
$$\mathbb{Z}^n/A(\mathbb{Z}^n) = \{\mathbf{q}_r + A(\mathbb{Z}^n) : r = 0, 1, \ldots, p-1\}.$$

The translation group G is induced by the lattice $\mathbb{Z}^n$. Although our results will be obtained with respect to this particular lattice only, our methods can be easily extended to all regular lattices, i.e. lattices of the form $C(\mathbb{Z}^n)$, where C

is an $n \times n$ invertible matrix. Following the tradition of all papers on Harmonic and Fourier analysis we give the definition of the Fourier transform on $L^1(\mathbb{R}^n)$:

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(\mathbf{t}) e^{-2\pi i \mathbf{t}\cdot\xi} d\mathbf{t}, \quad \xi \in \mathbb{R}^n.$$

We also reserve $\mathcal{F}$ to denote the Fourier transform on $L^2(\mathbb{R}^n)$. In addition, we adopt the notation $\mathbb{T}^n := [-1/2, 1/2)^n$ and δ_i ($i \in I \subseteq \mathbb{N}$) for the sequence defined by

$$\delta_i(l) := \begin{cases} 1 & \text{if} \quad l = i \\ 0 & \text{if} \quad l \neq i\ . \end{cases}$$

Before proceeding we need to include some final remarks on our notation. If A is a subset of a topological vector space, then $[A]$ denotes its linear span and A^- denotes the closure of A. Moreover, if B is a matrix (even an infinite one), then $[B]_i$ denotes the i-th column of B. We conclude this section with the characterization of the autocorrelation function of a set of frame generators of a shift-invariant subspace of $L^2(\mathbb{R}^n)$. i.e. of a set of functions $\{\phi_l : l \in I\}$ such that $\{T_{\mathbf{k}}\phi_l : l \in I\}$ is a frame for its closed linear span. The following lemma follows directly from Theorem 4 in [17].

Lemma 1.1. *Let $I \subseteq \mathbb{N}$ and $\{\phi_k : k \in I\}$ be a subset of $L^2(\mathbb{R}^n)$. Define*

$$a_{l,k}(\xi) := \sum_{\mathbf{m}\in\mathbb{Z}^n} \hat{\phi}_k(\xi + \mathbf{m})\overline{\hat{\phi}_l(\xi + \mathbf{m})} \quad k, l \in I, \quad \xi \in \mathbb{T}^n$$

and $a_k(\xi) := (a_{1,k}(\xi), a_{2,k}(\xi), \ldots)$.

(a) Assume that for every $k \in I$ the function $\xi \to \|a_k(\xi)\|_{\ell^2}$ is in $L^2(\mathbb{T}^n)$ and that the linear operators $\Phi(\xi)$ defined for a.e. $\xi \in \mathbb{T}^n$ on $[\delta_k : k \in I]$ by the equation $\Phi(\xi)\delta_k = a_k(\xi)$ satisfy the following properties:

1. *Φ belongs to $L^\infty(\mathbb{T}^n, \mathcal{B}(\ell^2(I)))$, i.e. Φ is weakly measurable and for a.e. $\xi \in \mathbb{T}^n$ the operator $\Phi(\xi)$ belongs to $\mathcal{B}(\ell^2(I))$ and $\|\Phi\|_\infty := \operatorname{essup}\{\|\Phi(\xi)\| : \xi \in \mathbb{T}^n\} < \infty$.*
2. *Let $P(\xi)$ be the range projection of $\Phi(\xi)$ a.e. There exists $B > 0$ such that for every $x \in P(\xi)(\ell^2(I))$ we have $B\|x\| \leq \|\Phi(\xi)x\|$.*

Then $\{T_{\mathbf{k}}\phi_l : l \in I, k \in \mathbb{Z}^n\}$ is a frame for its closed linear span with frame bounds B and $\|\Phi\|_\infty$.

Conversely, if $\{T_{\mathbf{k}}\phi_l : l \in I, k \in \mathbb{Z}^n\}$ is a frame for its closed linear span with frame constants B, C, then there exists $\Phi \in L^\infty(\mathbb{T}^n, \mathcal{B}(\ell^2(I)))$ such that $\|\Phi\|_\infty \leq C$ also satisfying

$$\Phi(\xi)_{l,k} = a_{l,k}(\xi) \quad k, l \in I, \quad a.e.\ \mathbb{T}^n$$

and property (2). Finally, $\{T_{\mathbf{k}}\phi_l : l \in I, k \in \mathbb{Z}^n\}$ is a Parseval frame for its closed linear span if and only if $\Phi(\xi)$ is for a.e. ξ an orthogonal projection.

Furthemore, $\{T_{\mathbf{k}}\phi_l : l \in I, k \in \mathbb{Z}^n\}$ *is a Riesz basis for its closed linear span if and only* $P(\xi) = I_{\ell^2(I)}$*, for a.e.* $\xi \in \mathbb{T}^n$*.*

The function Φ is also known as the Grammian of the set $\{\phi_l : l \in I\}$.

2. Radial FMRAs

In the present section we will develop the theory of singly generated GFMRAs of $L^2(\mathbb{R}^n)$ defined by radial frame scaling functions. We refer to these GFMRAs as *Radial FMRAs*. According to Lemma 1.1 the Fourier transform of a frame scaling functions which is not a Riesz scaling function cannot be continuous and at the same time satisfy even a mild decay condition. Indeed, if ϕ is any square-integrable function whose integer translates form a frame for the subspace they generate, then, according to Lemma 1.1, the linear operator $\Phi(\xi)$ is defined on $\mathbb{C}$, a.e. in $\mathbb{T}^n$, by

$$\Phi(\xi)z = z\left(\sum_{\mathbf{m}\in\mathbb{Z}^n}\left|\hat{\phi}(\xi+\mathbf{m})\right|^2\right), \qquad z \in \mathbb{C}\,.$$

It now follows that the range of $\Phi(\xi)$ is $\mathbb{C}$, for a.e. $\xi \in \mathbb{T}^n$ for which $\Phi(\xi) \neq 0$. By virtue of property (2) of Lemma 1.1, there exists $B > 0$ such that, for each such ξ, we have $B\,|z| \leq |\Phi(\xi)z|$, for every $z \in \mathbb{C}$. Since $\Phi(\xi)$ is defined on $\mathbb{C}$, we can identify the operator-valued function Φ with the 1-periodic function $\xi \to \sum_{\mathbf{m}\in\mathbb{Z}^n}\left|\hat{\phi}(\xi+\mathbf{m})\right|^2$ and from now on consider Φ as a complex-valued measurable function. The discussion on Φ now implies that, for a.e. $\xi \in \mathbb{T}^n$ satisfying $\Phi(\xi) > 0$ we have $\Phi(\xi) > B > 0$. So, if $\hat{\phi}$ is continuous and, for some $\epsilon > 0$ satisfies $\left|\hat{\phi}(\xi)\right| = O((1+\|\xi\|)^{-\frac{1}{2}-\epsilon})$, for every $\xi \in \mathbb{R}^n$, then Φ has to be continuous. Consequently $\Phi(\xi) > B > 0$ for every $\xi \in \mathbb{T}^n$. Thus, according to the last conclusion of Lemma 1.1, the integer translates of ϕ form a Riesz basis for the subspace they generate. The preceding discussion now leads to the conclusion that, single frame scaling functions which are not Riesz scaling functions cannot have a variety of forms, but as it has been shown in [18] this drawback can be rectified by using sets of frame multiscaling functions. In the present paper we will be exclusively using single frame scaling functions and, in particular, those whose Fourier transform is the characteristic function of a measurable subset of $\mathbb{R}^n$.

Our translation group is group-isomorphic to $\mathbb{Z}^n$, so, one can easily see that the regular representation of G on $\ell^2(G)$, defines a group, which in [18] is denoted by G^* and in our case is homeomorphically isomorphic to the discrete group $\mathbb{Z}^n$. Therefore, the dual group G^* is homeomorphically isomorphic to the n-dimensional torus $\mathbb{T}^n$. So, instead of using $\widehat{G^*}$ we use $\mathbb{T}^n$. Recall that we identified $\mathbb{T}^n$ with the product space $[-1/2, 1/2)^n$.

Now, let $\mathbb{D}$ be the sphere with radius $1/2$ centered at the origin, and ϕ be such that $\hat{\phi} := \chi_{\mathbb{D}}$. Since $\Phi(\xi) = \chi_{\mathbb{D}}(\xi)$, for every $\xi \in \mathbb{T}^n$ we have that $\{T_{\mathbf{k}}\phi : \mathbf{k} \in \mathbb{Z}^n\}$ is a Parseval frame for its closed linear span, which from now on we denote with V_0. *We will consider dilations induced by dilation matrices A satisfying the following property.*

Property D: There exists $c > 1$ such that for every $x \in \mathbb{R}^n$ we have $c\,\|x\| \leq \|Ax\|$.

Property D readily implies $\|A^{-1}\| \leq c^{-1} < 1$. However, it is interesting to note that Property D cannot be derived from the definition of dilation matrices. This fact can be demonstrated by the following example. Let

$$A = \begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}$$

Obviously A is invertible and leaves the integer lattice invariant, because all its entries are integers. However,

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -5 \\ 0 & 2 \end{pmatrix}.$$

Since one of the entries of A^{-1} has absolute value greater than 1 we get $\|A^{-1}\| > 1$, so A does not satisfy Property D.

Now, define $V_j := D^j(V_0)$, where $j \in \mathbb{Z}$. We will now establish $V_{-1} \subseteq V_0$. First, let $B := A^T$, where the superscript T denotes the transpose operation. Since $(A^T)^{-1} = (A^{-1})^T$ and the operator norm of a matrix is equal to the operator norm of its transpose, we obtain that dilation matrices A satisfying Property D, therefore, satisfies $\|B^{-1}\| < 1$. Thus $B^{-1}(\mathbb{D})$ is contained in $\mathbb{D}$. Next, let μ_0 be the measurable function defined on $\mathbb{R}^n$ such that $\mu_0(\xi) = \chi_{B^{-1}(\mathbb{D})}(\xi)$, for every $\xi \in \mathbb{T}^n$, which is periodically extended on $\mathbb{R}^n$ with respect to the tiling of $\mathbb{R}^n$ induced by the integer translates of $\mathbb{T}^n$. Then μ_0 belongs to $L^2(\mathbb{T}^n)$ and satisfies

$$\hat{\phi}(B\xi) = \mu_0(\xi)\hat{\phi}(\xi) \quad \text{a.e.},$$

because $\hat{\phi}(B\xi) = \chi_{B^{-1}(\mathbb{D})}(\xi)$, for every $\xi \in \mathbb{R}^n$. This implies that $D^*\phi$, belongs to V_0, which in turn establishes $V_{-1} \subseteq V_0$, and thus $V_j \subseteq V_{j+1}$, for every integer j. Since $\mathcal{F}(V_j) = L^2(B^j(\mathbb{D}))$, for all $j \in \mathbb{Z}$, we finally obtain that both properties in (b) of the definition of a GFMRA are satisfied. From the preceding argument we conclude that $\{V_j\}_j$ is a GFMRA of $L^2(\mathbb{R}^n)$, singly generated by the radial scaling function ϕ. So $\{V_j\}_j$ is a Radial FMRA of $L^2(\mathbb{R}^n)$. We may also occasionally refer to ϕ as a *Parseval frame scaling function* in order to indicate that $\{T_{\mathbf{k}}\phi : \mathbf{k} \in \mathbb{Z}^n\}$ is a Parseval frame for V_0. Following the terminology and the notation introduced in [18] the analysis operator S induced by the frame scaling set $\{\phi\}$ maps V_0 into $L^2(\mathbb{T}^n)$ and is defined by

$$Sf = \sum_{\mathbf{k}\in\mathbb{Z}} \langle f,\ T_{\mathbf{k}}\phi\rangle e_{\mathbf{k}},$$

where $e_{\mathbf{k}}(\xi) = e^{-2\pi i(\xi \cdot \mathbf{k})}$ for every $\xi \in \mathbb{R}^n$. Since ϕ is a Parseval frame scaling function we obtain that S is an isometry. Moreover it is not hard to verify that the range of S is the space containing all square-integrable $\mathbb{Z}^n$-periodic functions vanishing outside $\mathbb{D}$.

According to Definition 3 in [18], the low pass filter m_0 corresponding to ϕ is given by $m_0 := SD^*\phi$. Since we consider $\{V_j\}_j$ as singly generated we have only one low pass filter, so M_0, the low pass filter associated with the frame multiscaling set $\{\phi\}$ is equal to m_0. Since, S is an isometry we obtain $Y = S$, where Y is defined by the polar decomposition of S, namely $S = Y|S|$. In fact, we have $Y = S$[1]. Let us now find m_0. Taking the Fourier transforms of both sides of

$$D^*\phi = \sum_{\mathbf{k}\in\mathbb{Z}^n} \langle D^*\phi, T_{\mathbf{k}}\phi\rangle T_{\mathbf{k}}\phi \tag{3}$$

we obtain

$$\hat{\phi}(B\xi) = |detA|^{-1/2}\, m_0(\xi)\hat{\phi}(\xi) \qquad \text{a.e.} \tag{4}$$

Now, recall

$$\hat{\phi}(B\xi) = \mu_0(\xi)\hat{\phi}(\xi) \qquad \text{a.e.} \tag{5}$$

Unfortunately, the fact that, the set of the integer translates of ϕ is not a basis for V_0 but an overcomplete frame, does not automatically imply $|detA|^{1/2}\,\mu_0 = m_0$. However, both m_0 and μ_0 vanish outside $\mathbb{D}$, so eqs. (4) and (5) imply

$$m_0(\xi) = |detA|^{1/2}\,\chi_{B^{-1}(\mathbb{D})}(\xi), \quad \xi \in \mathbb{T}^n. \tag{6}$$

Obviously, all radial functions of the form $\chi_{\mathbb{D}}$, where $\mathbb{D}$ is a sphere centered at the origin with radius $r < 1/2$ are radial Parseval frame scaling functions. We will not distinguish this particular case from the case $r = 1/2$, because the latter case is generic and also optimizes the frequency spectrum subject to subband filtering, induced by this particular selection of the scaling function ϕ. This frequency spectrum is equal to the support of the autocorrelation function of ϕ, because every signal in V_0 will be encoded by the Analysis operator with an $\ell^2(\mathbb{Z})$-sequence, whose Fourier transform has support contained in $\mathbb{D}$. Therefore, the frequency spectrum subject to subband filtering induced by $\{V_j\}_j$ equals $\mathbb{D}$. This suggests that a prefiltering step transforming a random digital signal into another signal whose frequency spectrum is contained in $\mathbb{D}$ is necessary prior to the application of the decomposition algorithm induced by $\{V_j\}_j$. This prefiltering step is called initialization of the input signal. In the light of these remarks one might wonder whether we may be able to increase the frequency

[1]Following [18] $\tilde{m}_0 := YD^*\phi = m_0$; thus, $\tilde{M}_0 = \tilde{m}_0 = M_0$.

spectrum that these FMRAs can filter by allowing $r > 1/2$. We will later show that the selection $r = 1/2$ is optimum.

The frame scaling function can be determined in terms of Bessel functions, because it is a radial function.

$$\phi(R) = \frac{J_{\frac{n}{2}}(\pi R)}{(2R)^{\frac{n}{2}}}, \quad R > 0. \tag{7}$$

The proof of eq. (7) can be found in [19] Lemma 2.5.1.

We will not give any details regarding Bessel functions. However, the reader may refer to [19] and [3] for an extensive treatment of their main properties and of course to the bible of the topic [23]. Here, we only include the following formula.

$$J_a(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+a}}{k!\Gamma(k+a+1)} , \quad a > -1, \ x > 0.$$

The function J_a given by the above equation is called the *Bessel function of the first kind of order a.*

Apparently every function in V_0 is bandlimited, because its Fourier transform is supported on $\mathbb{D}$. Since $\mathbb{D}$ is contained in $\mathbb{T}^n$ we can readily infer from the classical sampling theorem that if f is in V_0, then

$$f = \sum_{\mathbf{k} \in \mathbb{Z}^n} f(\mathbf{k}) T_{\mathbf{k}} \omega , \tag{8}$$

where the RHS of the previous equation converges in the L^2-norm and $\omega(x_1, x_2, \ldots, x_n) = \prod_{q=1}^n \frac{\sin(\pi x_q)}{\pi x_q}$. If P_0 is the projection onto V_0, then applying P_0 on both sides of eq. (8) gives

$$f = \sum_{\mathbf{k} \in \mathbb{Z}^n} f(\mathbf{k}) P_0(T_{\mathbf{k}} \omega) = \sum_{\mathbf{k} \in \mathbb{Z}^n} f(\mathbf{k}) T_{\mathbf{k}} P_0(\omega) ,$$

because P_0 commutes with the translation operator $T_{\mathbf{k}}$, for every $\mathbf{k} \in \mathbb{Z}^n$. Since $P_0(\omega) = \phi$, we conclude the following sampling theorem:

Theorem 3. *Let f be in V_0. Then,*

$$f = \sum_{\mathbf{k} \in \mathbb{Z}^n} f(\mathbf{k}) T_{\mathbf{k}} \phi , \tag{9}$$

where the RHS of equation (9) converges in the L^2 norm. Moreover, the same series converges uniformly to f, if we assume that f is continuous.

Proof. The first conclusion of the theorem has already been established. We will now prove the uniform convergence to f of the series in the RHS of eq. (9)

assuming that f is continuous. Let $\mathbf{t} \in \mathbb{R}^n$. Then, for $N \in \mathbb{N}$, we have

$$\left| f(\mathbf{t}) - \textstyle\sum_{\|\mathbf{k}\|_\infty \le N} f(\mathbf{k}) T_{\mathbf{k}} \phi(\mathbf{t}) \right| \tag{10}$$

$$= \left| \int_{\mathbb{T}^n} \left(\hat{f}(\xi) - \textstyle\sum_{\|\mathbf{k}\|_\infty \le N} f(\mathbf{k}) e^{-2\pi i \xi \cdot \mathbf{k}} \chi_{\mathbb{D}}(\xi) \right) e^{2\pi i \xi \cdot \mathbf{t}} d\mathbf{t} \right| \tag{11}$$

$$\le \left\| \hat{f} - \textstyle\sum_{\|\mathbf{k}\|_\infty \le N} f(\mathbf{k}) e_{\mathbf{k}} \chi_{\mathbb{D}} \right\|_2 . \tag{12}$$

As $N \to \infty$ the first term of the RHS of the previous inequality tends to zero. This establishes the final conclusion of theorem 3. □

Remark 1. The continuity hypothesis that we imposed on f in order to derive the uniform convergence to f of the series in the RHS of eq. (9) is not at all artificial. It is well known that since f is band-limited, f is almost everywhere equal to an infinitely differentiable function, namely the inverse Fourier transform of $\hat{f}$. Thus instead of using f itself we can use the reflection of $\mathcal{F}(\mathcal{F}(f))$.

Remark 2. Although ϕ is a radial function, its dilations $D^j \phi$, for $j \neq 0$ may cease to be radial, for if $j = -1$, then $\mathcal{F}(D^* \phi) = |\det A|^{1/2} \chi_{B^{-1}(\mathbb{D})}$ and $B^{-1}(\mathbb{D})$ may not be an isotropic domain. However, in several interesting cases of dilation matrices A all the dilations of ϕ are radial.

The preceding remark motivates the following definition:

Definition 4. *An expansive matrix A is called radially expansive if $A = aU$, where $a > 0$ and U is a unitary matrix.*

Expansive matrices obviously satisfy $a^n = |detA|$ and $\|A\| = a$. Apparently radially expansive dilation matrices satisfy Property D. When this is the case, we immediately obtain that all $D^j \phi$ are radial functions as well, and, in particular,

$$(D^{-1}\phi)(R) = \frac{J_{\frac{n}{2}}(\pi a^{-1} R)}{(2R)^{\frac{n}{2}}}, \quad R > 0. \tag{13}$$

Combining eqs. (4), (6) and (13) we conclude

$$\widehat{m_0}(\mathbf{k}) = \frac{J_{\frac{n}{2}}(\pi a^{-1} \|\mathbf{k}\|)}{(2\|\mathbf{k}\|)^{\frac{n}{2}}}, \quad \mathbf{k} \in \mathbb{Z}^n.$$

Proposition 5. *Let A be a radially expansive dilation matrix, and D_r be the sphere with radius r centered at the origin. Then, there exists $r_0 > 0$ such that, if $r > r_0$ and $\phi := \mathcal{F}^{-1}(\chi_{D_r})$, then no measurable $\mathbb{Z}^n$-periodic function μ satisfies*

$$\hat{\phi}(B\xi) = \mu(\xi)\hat{\phi}(\xi) \tag{14}$$

for a.e. ξ in $\mathbb{R}^n$. Thus, such a ϕ cannot be a frame scaling function.

Proof. Let $r > 0$ and $\hat{\phi} = \chi_{D_r}$. Assume $A = aU$, where $a > 1$. Then, $\hat{\phi}(B\xi) = \chi_{D_{r/a}}(\xi)$, a.e. ξ in $\mathbb{R}^n$, which in conjunction with eq. (14) imply $\mu(\xi) = \chi_{D_{r/a}}(\xi)$ for a.e. $\xi \in \mathbb{T}^n$. If $\frac{r}{a} \geq \frac{1}{\sqrt{2}}$, then $\mu(\xi) = 1$ for a.e. ξ in $\mathbb{T}^n$, which, due to the $\mathbb{Z}^n$-periodicity of μ, implies $\mu(\xi) = 1$ a.e. in $\mathbb{R}^n$. This obviously contradicts eq. (14). Thus, $r < \frac{a}{\sqrt{2}}$. Now, pick such an r. If $r \leq r_0 := \frac{a}{a+1}$, then D_r and $\mathbf{k}+D_{r/a}$, for every $\mathbf{k} \in \mathbb{Z}\backslash\{0\}$ do not intersect.

Now, assume $r > r_0$. In this case we obviously have $\frac{1}{2} < r_0 < r < \frac{a}{\sqrt{2}}$. Next, translate $\mathbb{T}^n$ by $\mathbf{u} := (1, 0, 0, \ldots, 0)$. Due to the periodicity of μ we have $\mu(\xi) = 1$ for a.e. ξ in the intersection of the sphere $\mathbf{u} + D_{r/a}$ and $\mathbf{u} + \mathbb{T}^n$. Since $r > r_0$ we can find x such that $\max\{\frac{r}{a}, \frac{1}{2}\} < x < r_0$. Then, there exists a ball centered at $(x, 0, 0, \ldots, 0)$, which is contained in the intersection of $\mathbf{u} + D_{r/a}$, $\mathbf{u} + \mathbb{T}^n$ and $\mathbb{R}^n\backslash D_r$, so, eq. (14) fails to be true for every point in this ball. $\square$

If $1/2 < r \leq r_0$, then $\phi := \mathcal{F}^{-1}(\chi_{D_r})$ is a frame scaling function. This can be shown by invoking lemma 1.1, which establishes that $\{T_{\mathbf{k}}\phi : \mathbf{k} \in \mathbb{Z}^n\}$ is a frame (but not a Parseval frame) for V_0 and the argument showing. $V_j = \mathcal{F}^{-1}(L^2(B^j(\mathbb{D}))$. We will omit the details of this proof since we think that this particular case is not as interesting as the case $r \leq 1/2$, because the FMRAs defined by such frame scaling functions ϕ still cannot filter the entire n-dimensional torus $\mathbb{T}^n$. Having finished this intermezzo we return to the initial hypothesis, $r = 1/2$.

Let us now discuss the construction of certain frame multiwavelet sets associated with $\{V_j\}_j$. It has been pointed in [18] that unlikely with MRAs the cardinality of the frame multiwavelet sets associated with the same GFMRA may vary. This observation readily indicates that there is room for alternative constructions of GFMRA frame multiwavelet sets. However, all these sets must satisfy certain necessary and sufficient conditions, which we present in theorem 6.

In the discussion that follows we present two constructions of frame multiwavelet sets associated with $\{V_j\}_j$. Each one has its own merit. The first one of them does not depend on the dimension of the underlying Euclidean space $\mathbb{R}^n$, and we believe that it is the most elegant from all of them. The second one specifically applies only if the underlying space is $\mathbb{R}^2$ and the dilation operators are defined by $A = 2I_2$ or $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. It can easily be seen that both matrices are radially expansive dilation matrices. The latter of these two matrices generates the so-called Quincunx subsampling lattice. Subsampling lattices are used in the applications of the Decomposition and Reconstruction algorithms. We will discuss Decomposition and Reconstruction algorithms induced by $\{V_j\}_j$ and initialization algorithms in a follow up paper.

First construction: We adopt the proof of Theorem 13 of [18] to the radial FMRA $\{V_j\}_j$. First, set $\hat{V}_j := \mathcal{F}(V_j)$ and $\hat{W}_j := \mathcal{F}(W_j)$, where $j \in \mathbb{Z}$. Recall that $\hat{V}_0 = \mathcal{F}(V_0) = L^2(\mathbb{D})$, and that the Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$. Combining these facts with $\hat{V}_{-1} = L^2(B^{-1}(\mathbb{D}))$, we conclude

$$\hat{W}_{-1} = \hat{V}_0 \cap \hat{V}_{-1}^{\perp} = L^2(\mathcal{Q}),$$

where $\mathcal{Q}$ is the annulus $\mathbb{D} \cap (B^{-1}(\mathbb{D}))^c$, and the superscript c denotes the set-theoretic complement. Since an arbitrary orthogonal projection R defined on a Hilbert space H maps every orthonormal basis of H onto a Parseval frame for $R(H)$ ([2, 12]), we obtain that the orthogonal projection defined on $L^2(\mathbb{T}^n)$ by multiplication with the indicator function of $\mathcal{Q}$ gives a Parseval frame for $L^2(\mathcal{Q})$, namely the set $\{e_{\mathbf{k}}\chi_{\mathcal{Q}} : \mathbf{k} \in \mathbb{Z}^n\}$.

Next, observe that each $\mathbf{k} \in \mathbb{Z}^n$ belongs to exactly one of the elements of the quotient group $\mathbb{Z}^n/A(\mathbb{Z}^n)$; thus there exist $\mathbf{q}$ and $r \in \{0, 1, \ldots, p-1\}$ such that $\mathbf{k} = \mathbf{q}_r + A(\mathbf{q})$. Therefore, $e_{\mathbf{k}} = e_{\mathbf{q}_r} e_{A(\mathbf{q})}$. We now define the following functions:

$$h_r := e_{\mathbf{q}_r}\chi_{\mathcal{Q}} \quad r \in \{0, 1, \ldots, p-1\}. \tag{15}$$

Apparently $\{e_{A(\mathbf{k})}h_r : \mathbf{k} \in \mathbb{Z}^n,\ r = 0, 1, \ldots, p-1\}$ is a Parseval frame for $L^2(\mathcal{Q})$, thus for $\hat{W}_{-1}$ as well. Therefore, $\{T_{A(\mathbf{k})}\mathcal{F}^{-1}h_r : \mathbf{k} \in \mathbb{Z}^n,\ r = 0, 1, \ldots, p-1\}$ is a Parseval frame for W_{-1}, because the Fourier transform is unitary. Setting $\psi_r := D\mathcal{F}^{-1}h_r$ $(r = 0, 1, \ldots, p-1)$ we finally have that $\{T_{\mathbf{k}}\psi_r : \mathbf{k} \in \mathbb{Z}^n,\ r = 0, 1, \ldots, p-1\}$ is a Parseval frame for W_0, therefore $\{\psi_r : r = 0, 1, \ldots, p-1\}$ is a Parseval frame multiwavelet set associated with the FMRA $\{V_j\}_j$. This concludes the first construction of a frame multiwavelet set associated with $\{V_j\}_j$.

The reader might wonder whether it is possible to give a more explicit formula for the frame wavelets ψ_r. In the light of remark 2, ψ_0 may not be radial as well. This may yield a rather unattractive time domain formula for all these wavelets. It worths mentioning that ψ_r, where $r > 0$, are never radial if ψ_0 is radial. However, if A is a radially expansive dilation matrix and $a = \|A\|$, then

$$(\mathcal{F}^{-1}h_0)(R) = \frac{J_{\frac{n}{2}}(\pi R)}{(2R)^{\frac{n}{2}}} - \frac{J_{\frac{n}{2}}(\pi \frac{R}{a})}{(2aR)^{\frac{n}{2}}}\,, \quad R > 0.$$

Therefore, under this assumption, ψ_0 is radial and

$$\psi_0(R) = \frac{a^{\frac{n}{2}} J_{\frac{n}{2}}(\pi a R) - J_{\frac{n}{2}}(\pi R)}{(2aR)^{\frac{n}{2}}}\,, \quad R > 0;$$

and for $r = 1, 2, \ldots, p-1$.

$$\begin{aligned}\psi_r(t) &= DT_{\mathbf{q}_r}D^*\psi_0(t) = \psi_0(t - A^{-1}\mathbf{q}_r)\\ &= \frac{a^{\frac{n}{2}} J_{\frac{n}{2}}(\pi a \|t - A^{-1}\mathbf{q}_r\|) - J_{\frac{n}{2}}(\pi \|t - A^{-1}\mathbf{q}_r\|)}{(2a\|t - A^{-1}\mathbf{q}_r\|)^{\frac{n}{2}}}, \quad t \in \mathbb{R}^n.\end{aligned}$$

Notice that in this case $p = |detA| = a^n$.

We now continue with the preliminaries of the second construction. From now on and until the end of the present section we work with GFMRAs of $L^2(\mathbb{R}^2)$ only.

One of the instrumental tools of this construction is the square root of the autocorrelation function Φ, which is defined by $A(\xi)^2 = \Phi(\xi)$, a.e. on $\mathbb{T}^2$ ([18]). Also, the inverse of $A(\xi)$ is defined on the range of $\Phi(\xi)$ and is denoted by $A(\xi)^{-1}$. It has also been proved in [18] that the range projection P of the Analysis operator S is defined by $P\omega(\xi) = P(\xi)\omega(\xi)$, where $\omega \in L^2(\mathbb{T}^2)$, and that for a.e. $\xi \in \mathbb{T}^2$ the range projection of $\Phi(\xi)$ is the projection $P(\xi)$. For the sake of completeness, it must be noted that $P(\cdot)$ is a projection-valued weakly measurable function defined on $\mathbb{T}^2$. Since, $\Phi = \chi_{\mathbb{D}}$, we deduce $P(\xi) = \chi_{\mathbb{D}}(\xi)$ a.e. in $\mathbb{T}^2$. The latter observation in conjunction with the preceding argument imply $A(\xi)^{-1} = 1$, if $\xi \in \mathbb{D}$. For all other $\xi \in \mathbb{T}^2$ we have $A(\xi) = 0$, so for these ξ we adopt the notational convention $A(\xi)^{-1} = 0$. Last but not least, an abelian group very instrumental in the discussion that follows is the kernel of the homomorphism ρ defined by

$$\rho(\xi)(\mathbf{k}) = e^{2\pi i(\xi \cdot A\mathbf{k})}, \quad \mathbf{k} \in \mathbb{Z}^2.$$

The latter equation implies that, for every $\xi \in \mathbb{T}^2$, $\rho(\xi)$ is the unique point in $\mathbb{T}^2$, such that $\rho(\xi) + \mathbf{k} = A^T\xi$. The kernel of ρ is homeomorphically isomorphic to dual group of the quotient group $\mathbb{Z}^2/A(\mathbb{Z}^2)$ ([18]). Now, let us fix $\mathbf{k}_r$, where $r = 0, 1, \ldots, p-1$, in $\mathbb{T}^2$, so that $\text{Ker}\rho = \{\mathbf{k}_r : r = 0, 1, \ldots, p-1\}$.

We are now ready to adopt Theorem 10 from [18] for case of the FMRA $\{V_j\}_j$ generated by ϕ.

Theorem 6. *Let $I \subseteq \mathbb{N}$. Assume $\tilde{H} : \mathbb{T}^2 \to \mathcal{B}(\ell^2(I), \mathbb{C})$. Define*

$$\tilde{Q}_2(\xi) := \sum_{r=0}^{p-1} \tilde{H}(\xi + \mathbf{k}_r)^* \tilde{H}(\xi + \mathbf{k}_r).$$

Moreover assume that the following conditions are satisfied

(a) *For a.e. $\xi \in$ $suppP_2$, where $P_2(\xi)$ is the range projection of the operator $\tilde{Q}_2(\xi)$, the operator $\tilde{Q}_2(\xi) \mid_{P_2(\xi)(\ell^2(I))}$: $P_2(\xi)(\ell^2(I)) \to P_2(\xi)(\ell^2(I))$ is invertible and the functions $\xi \to \left\| \tilde{Q}_2(\xi) \mid_{P_2(\xi)(\ell^2(I))} \right\|$, $\xi \to \left\| (\tilde{Q}_2(\xi) \mid_{P_2(\xi)(\ell^2(I))})^{-1} \right\|$ are essentially bounded.*

(b) *For a.e. $\xi \in \mathbb{T}^2$ the closed linear span of the columns of the matrix*

$$\begin{pmatrix} M_0(\xi) & \tilde{H}(\xi) \\ M_0(\xi + \mathbf{k}_1) & \tilde{H}(\xi + \mathbf{k}_1) \\ \vdots & \vdots \\ M_0(\xi + \mathbf{k}_{p-1}) & \tilde{H}(\xi + \mathbf{k}_{p-1}) \end{pmatrix}$$

is equal to $\tilde{P}(\xi)(\mathbb{C}^p)$, *where*

$$\tilde{P}(\xi) = \sum_{r=0}^{p-1} \bigoplus P(\xi + \mathbf{k}_r) \quad a.e.\ in\ \mathbb{T}^2\ and$$

(c) $0 = \sum_{r=0}^{p-1} M_0(\xi + \mathbf{k}_r)^* \tilde{H}(\xi + \mathbf{k}_r)$ *a.e.*

If we define

$$\psi_i := \sum_{m,n\in\mathbb{Z}} a_{m,n}^{(i)} D T_1^m T_2^n \phi,$$

where $\{a_{m,n}^{(i)} : i \in I, m, n \in \mathbb{Z}\}$ *are defined by the equation*

$$[\tilde{H}(\cdot)]_i = \sum_{m,n\in\mathbb{Z}} a_{m,n}^{(i)} e_{m,n}$$

then, $\{\psi_i : i \in I\}$ *is a frame multiwavelet set associated with the FMRA* $\{V_j\}_j$.

A measurable, $\mathbb{Z}^2$-periodic operator-valued function $\tilde{H}$, satisfying the hypotheses of the previous theorem is called *a high pass filter associated with* M_0. If the dilation matrix satisfies Property D, then one obvious choice for $\tilde{H}$ following from eq. (15) is

$$\tilde{H} = (e_{\mathbf{q}_0}\chi_Q, e_{\mathbf{q}_1}\chi_Q, \ldots, e_{\mathbf{q}_{p-1}}\chi_Q)\ .$$

Let us first study the case where the dilation matrix $A = 2I_2$. In this case it is well-known that $p = 4$ and, that we can set $\mathbf{k}_1 = (\frac{1}{2}, 0)$, $\mathbf{k}_2 = (\frac{1}{2}, \frac{1}{2})$ and $\mathbf{k}_3 = (0, \frac{1}{2})$. The reader will find useful to recall, that addition in $\mathbb{T}^2$ is defined modulo the integer lattice $\mathbb{Z}^2$.

Obviously,

$$\tilde{P}(\xi) = \begin{pmatrix} \chi_{\mathbb{D}}(\xi) & 0 & 0 & 0 \\ 0 & \chi_{\mathbb{D}+\mathbf{k}_1}(\xi) & 0 & 0 \\ 0 & 0 & \chi_{\mathbb{D}+\mathbf{k}_2}(\xi) & 0 \\ 0 & 0 & 0 & \chi_{\mathbb{D}+\mathbf{k}_3}(\xi) \end{pmatrix}.$$

On the other hand, according to theorem 6, we must first determine the values of $\tilde{P}$ before finding the high pass filter $\tilde{H}$. All the values of $\tilde{P}$ are 4×4 diagonal matrices whose diagonal entries are either equal to 1 or 0. Therefore, the range of $\tilde{P}$ is finite.

So, we can find a partition of $\mathbb{T}^2$, say $\{B_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}\}$, where $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ is the vector formed by the entries of the main diagonal of an arbitrary value of $\tilde{P}$. Apparently each ϵ_p, where $p = 0, 1, 2, 3$, takes only two values, namely 0 and 1. Since each of the sets $\mathbb{D} + \mathbf{k}_r$, where $r = 0, 1, 2, 3$, overlap with at least another one of these sets, there will be no values of $\tilde{P}$ with a single non zero diagonal entry (see fig. 1). The definition of the addition operation on $\mathbb{T}^2$ implies that

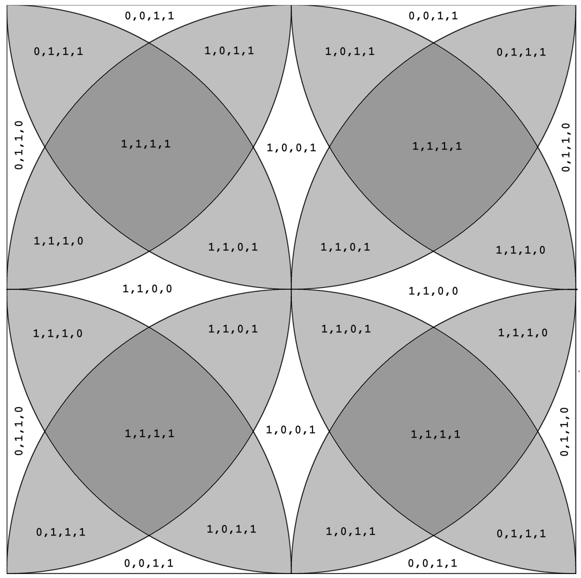

FIGURE 1

$\mathbb{D} + \mathbf{k}_1$ is the union of the two half disks with radii $1/2$ centered at $\mathbf{k}_1$ and $-\mathbf{k}_1$; $\mathbb{D} + \mathbf{k}_3$ is the union of the two half disks with radii $1/2$ centered at $\mathbf{k}_3$ and $-\mathbf{k}_3$; and, $\mathbb{D} + \mathbf{k}_2$ is the union of the four quarter disks with radii $1/2$ centered at $\mathbf{k}_2$, $-\mathbf{k}_2$, $(-\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$. Since all four sets $\mathbb{D} + \mathbf{k}_r$, where $r = 0, 1, 2, 3$, are symmetric with respect to both coordinate axes, it follows that all sets $B_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}$ share the same symmetry property. This observation contributes a great deal in identifying these sets. The reader can now refer to figure 1 where the subregions of $\mathbb{T}^2$ corresponding to each one of the vectors $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ are depicted. According to theorem 6 the values of the high pass filter $\tilde{H}$ must be row matrices. This is justified by the fact that $\{V_j\}_j$ is generated by a single scaling function. However, the range of every $\tilde{P}(\xi)$ is a subspace of $\mathbb{C}^4$. Furthermore, according to hypothesis (b) of theorem 6 the columns of the modulation matrix must span $\tilde{P}(\xi)(\mathbb{C}^4)$. Thus, we anticipate that the modulation matrix must have at least three more columns. So, $\tilde{H}(\xi)$ must be at least 1×3 matrix. For reasons that will become more clear later on in our discussion we choose $\tilde{H}(\xi)$ to be 1×4 matrix, for every $\xi \in \mathbb{T}^2$, namely $\tilde{H}(\xi) := 2(\tilde{h}_1(\xi), \tilde{h}_2(\xi), \tilde{h}_3(\xi), \tilde{h}_4(\xi))$. The factor 2 in the RHS of the previous equation is a normalization factor that helps to obtain a simple form for the each of the functions $\tilde{h}_i$. According to the conclusion of theorem 6, the columns of $\tilde{H}$, i.e. the functions $\tilde{h}_i$ $(i = 1, 2.3, 4)$, define a frame multiwavelet set associated with $\{V_j\}_j$.

Therefore the modulation matrix has the following form:

$$\begin{pmatrix} \chi_{\frac{\mathbb{D}}{2}}(\xi) & \tilde{h}_1(\xi) & \tilde{h}_2(\xi) & \tilde{h}_3(\xi) & \tilde{h}_4(\xi) \\ \chi_{\frac{\mathbb{D}}{2}+\mathbf{k}_1}(\xi) & \tilde{h}_1(\xi+\mathbf{k}_1) & \tilde{h}_2(\xi+\mathbf{k}_1) & \tilde{h}_3(\xi+\mathbf{k}_1) & \tilde{h}_4(\xi+\mathbf{k}_1) \\ \chi_{\frac{\mathbb{D}}{2}+\mathbf{k}_2}(\xi) & \tilde{h}_1(\xi+\mathbf{k}_2) & \tilde{h}_2(\xi+\mathbf{k}_2) & \tilde{h}_3(\xi+\mathbf{k}_2) & \tilde{h}_4(\xi+\mathbf{k}_2) \\ \chi_{\frac{\mathbb{D}}{2}+\mathbf{k}_3}(\xi) & \tilde{h}_1(\xi+\mathbf{k}_3) & \tilde{h}_2(\xi+\mathbf{k}_3) & \tilde{h}_3(\xi+\mathbf{k}_3) & \tilde{h}_4(\xi+\mathbf{k}_3) \end{pmatrix} \quad \text{a.e. in } \mathbb{T}^2.$$

The disk $\frac{\mathbb{D}}{2}$ has radius $1/4$, so this disk and all its translations by $\mathbf{k}_r$ $(r = 1, 2, 3)$ have null intersections. Thus, for every ξ in $\mathbb{T}^2$, the first column of the modulation matrix has at most one non zero entry. Since, for every ξ in $\mathbb{T}^2$, the columns of the modulation matrix must span $\tilde{P}(\xi)(\mathbb{C}^4)$, we obtain the remaining columns of the modulation matrix, so that together with the first column they form the standard orthonormal basis of $\tilde{P}(\xi)(\mathbb{C}^4)$. This suggests that the high pass filters h_i are the $\mathbb{Z}^2$-periodic extensions of the characteristic functions of certain measurable subsets of $\mathbb{T}^2$. Next, we will identify those subsets of $\mathbb{T}^2$, which we will denote by C_i, where $i = 1, 2, 3, 4$.

Remark 3. Let Q be the first quadrant of T^2. Then it is not difficult to verify that the family $\{Q + \mathbf{k}_r : r = 0, 1, 2, 3\}$ forms a partition of $\mathbb{T}^2$, in the sense that $\mathbb{T}^2 = \cup_{r=0}^{3} Q + \mathbf{k}_r$, but the intersections of every two of the sets $Q + \mathbf{k}_r$ have zero measure. Now, let ξ be in $Q + \mathbf{k}_r$. Then $\xi = \xi_0 + \mathbf{k}_r$, where $\xi_0 \in Q$. Without any loss of generality we can assume $r = 1$. Then,

$$(\tilde{h}_i(\xi), \tilde{h}_i(\xi+\mathbf{k}_1), \tilde{h}_i(\xi+\mathbf{k}_2), \tilde{h}_i(\xi+\mathbf{k}_3))^T = (\tilde{h}_i(\xi_0+\mathbf{k}_1), \tilde{h}_i(\xi_0), \tilde{h}_i(\xi_0+\mathbf{k}_3), \tilde{h}_i(\xi_0+\mathbf{k}_2))^T .$$

Thus, the values of the modulation matrix are completely determined by its values, when ξ ranges only throughout the first quadrant.

As we have previously mentioned, the family $\{B_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}\}$, where $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ ranges throughout the vectors formed by the diagonal entries of the values of $\tilde{P}$, is a partition of $\mathbb{T}^2$. Therefore, $\{Q \cap B_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}\}$ is a partition of Q. Furtheremore, each of the sets $Q \cap B_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}$ is, in turn, partitioned into a finite number of subsets which are formed by the intersections of $Q \cap B_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}$ with each one of the disks $\frac{\mathbb{D}}{2}$, $\frac{\mathbb{D}}{2} + \mathbf{k}_1$, $\frac{\mathbb{D}}{2} + \mathbf{k}_2$, $\frac{\mathbb{D}}{2} + \mathbf{k}_3$ and the complement of their union. This results in a partition of Q into 29 sets (see fig. 2). We denote these sets by E_s, where $1 \leq s \leq 29$. We now have to obtain, for every ξ in each of the sets E_s, the remaining four columns of the modulation matrix, so that they span $\tilde{P}(\xi)(\mathbb{C}^4)$. This process is not difficult to carry out. However, for the sake of the clarity of our presentation, we deem necessary to show how to specifically accomplish this task, when ξ belongs to three of the sets E_s.

Case $s = 1$. This set is contained in the complement of the union of the disks $\frac{\mathbb{D}}{2} + \mathbf{k}_r$, $r = 0, 1, 2, 3$, so the first column of the modulation matrix at ξ is equal

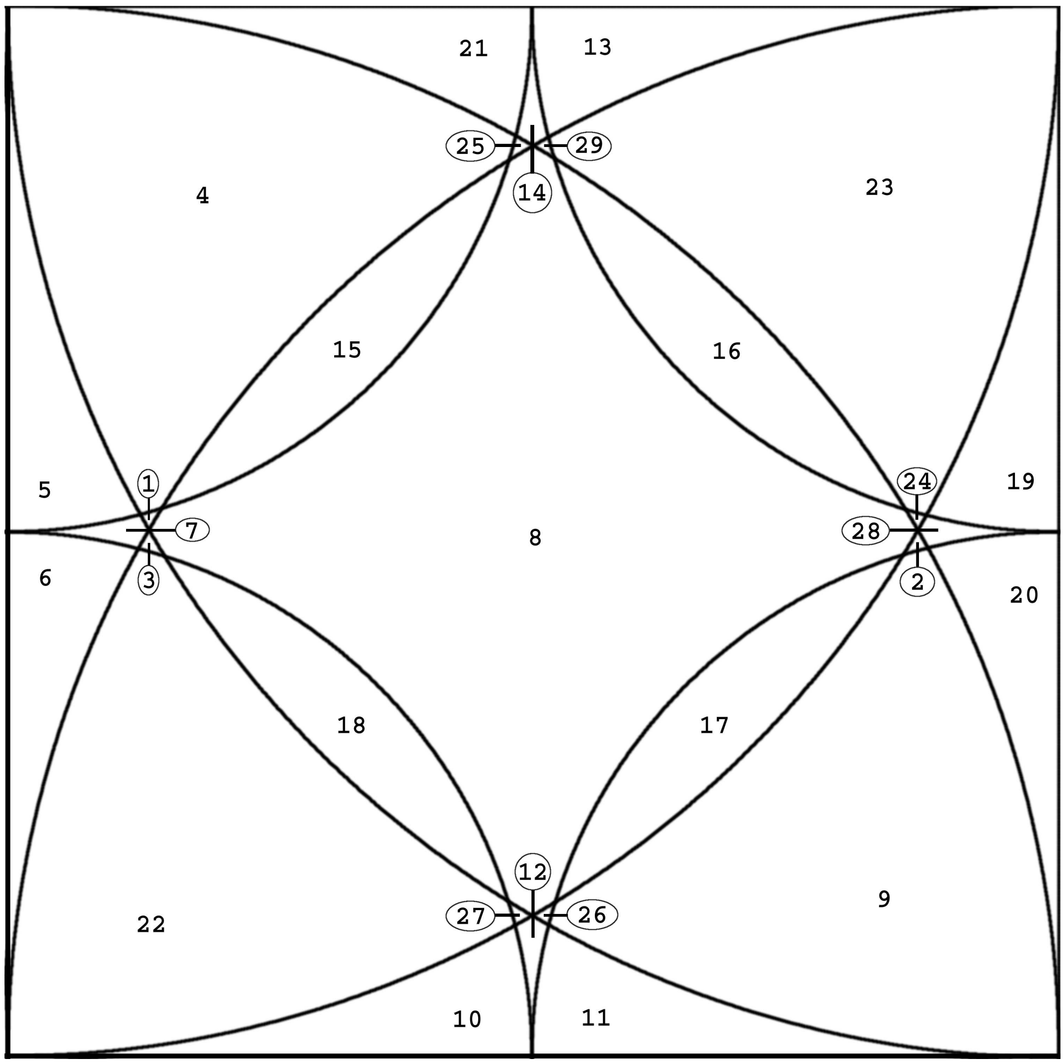

FIGURE 2

to zero. Now, let $\xi \in E_1$. On the other hand, $\tilde{P}(\xi)(\mathbb{C}^4) = \mathbb{C} \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C}$, so we choose to complement the modulation matrix by setting its second, third and fourth columns equal to $(1,0,0,0)^T$, $(0,0,1,0)^T$ and $(0,0,0,1)^T$ respectively, and the fifth column equal to zero.

Case $s = 18$. Let $\xi \in E_{18}$. Then, $\tilde{P}(\xi)(\mathbb{C}^4) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Moreover, $\xi \in \frac{\mathbb{D}}{2}$. This suggests the following form for the modulation matrix at ξ:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

Case $s = 19$. Let $\xi \in E_{19}$. Then, $\tilde{P}(\xi)(\mathbb{C}^4) = 0 \oplus \mathbb{C} \oplus \mathbb{C} \oplus 0$. Now, $\xi \in (\frac{\mathbb{D}}{2} + \mathbf{k}_2)$ yielding the following form for the modulation matrix at ξ:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

It is now easy to verify that C_i, where $i = 1, 2, 3, 4$, are the sets depicted in figures 3, 4, 5 and 6 respectively.

Let us now briefly review the case $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. In this case $p = |\det A| = 2$. It is also not hard to verify $\mathbf{k}_1 = (\frac{1}{2}, \frac{1}{2})$ and

$$\tilde{P}(\xi) = \begin{pmatrix} \chi_{\mathbb{D}}(\xi) & 0 \\ 0 & \chi_{\mathbb{D}+\mathbf{k}_1}(\xi) \end{pmatrix} .$$

Notice that $\mathbb{D} + \mathbf{k}_1$ is now the union of the four quarter disks with radii $\frac{1}{2}$ centered at the vertices of the fundamental domain $\mathbb{T}^2$.

Each one of them overlaps with $\mathbb{D}$ (fig. 7(a)). This, as in the case of $A = 2I_2$, yields a partition of $\mathbb{T}^2$, namely the collection of subsets $B_{(\epsilon_0,\epsilon_1)}$, where, (ϵ_0, ϵ_1) is the vector of the entries of the main diagonal of an arbitrary value of $\tilde{P}$, and $B_{(\epsilon_0,\epsilon_1)}$ contains all points in $\mathbb{T}^2$ at which the vector of the entries of the main diagonal of $\tilde{P}$ is equal to (ϵ_0, ϵ_1) (see fig. 7(b)). The low pass filter is now given by $m_0(\xi) = \sqrt{2}\chi_{\frac{\mathbb{D}}{\sqrt{2}}}(\xi)$, $\xi \in \mathbb{T}^2$, (see eq. (6)). This can easily follow from the form of the dilation matrix which is a composition of a rotation by $\pi/4$ matrix and $\sqrt{2}I_2$. We can now take $\tilde{H}(\xi) := \sqrt{2}(\tilde{h}_1(\xi), \tilde{h}_2(\xi))$, for every $\xi \in \mathbb{T}^2$. The

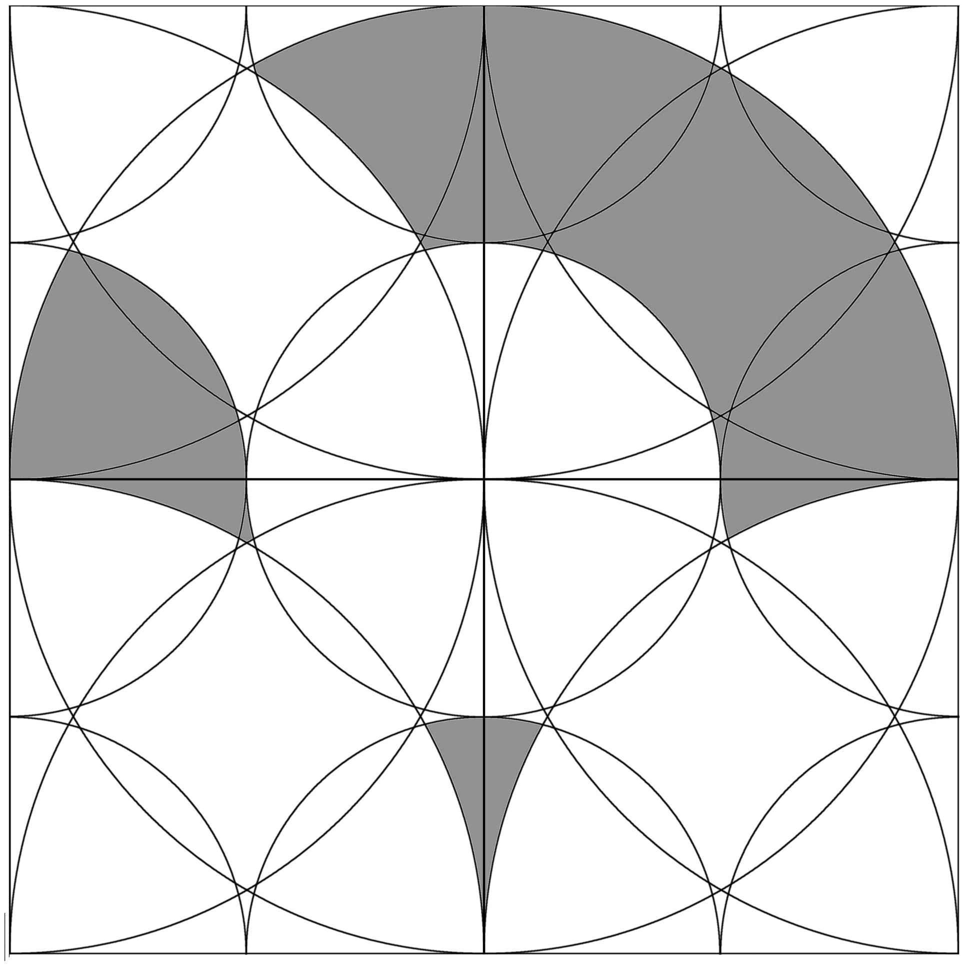

FIGURE 3

modulation matrix now has a much simpler form, namely

$$\begin{pmatrix} \chi_{\frac{\mathbb{D}}{\sqrt{2}}}(\xi) & \tilde{h}_1(\xi) & \tilde{h}_2(\xi) \\ \chi_{\frac{\mathbb{D}}{\sqrt{2}}+\mathbf{k}_1}(\xi) & \tilde{h}_1(\xi+\mathbf{k}_1) & \tilde{h}_2(\xi+\mathbf{k}_1) \end{pmatrix} \quad \text{a.e. in } \mathbb{T}^2.$$

Let us now set Q to be the closed square whose vertices are the mid points of the sides of $\mathbb{T}^2$. It is not hard to see that $Q+\mathbf{k}_1$ is the union of the four orthogonal isosceles triangles defined by the vertices of Q and $\mathbb{T}^2$. Obviously, $\{Q, Q+\mathbf{k}_1\}$ is a partition of $\mathbb{T}^2$ modulo null sets. An argument similar to the one in remark 3 shows that it is enough to determine the filters $\tilde{h}_i$ $(i=1,2)$ only on Q. It will also be helpful to observe that the sides of Q are tangent to the circle of radius $\frac{\sqrt{2}}{2}$ centered at the origin and that Q can also be partitioned by the sets $Q \cap B_{(\epsilon_0,\epsilon_1)}$, where $(\epsilon_0,\epsilon_1)=(1,0),(1,1)$ (see fig. 7(b)). Each of these

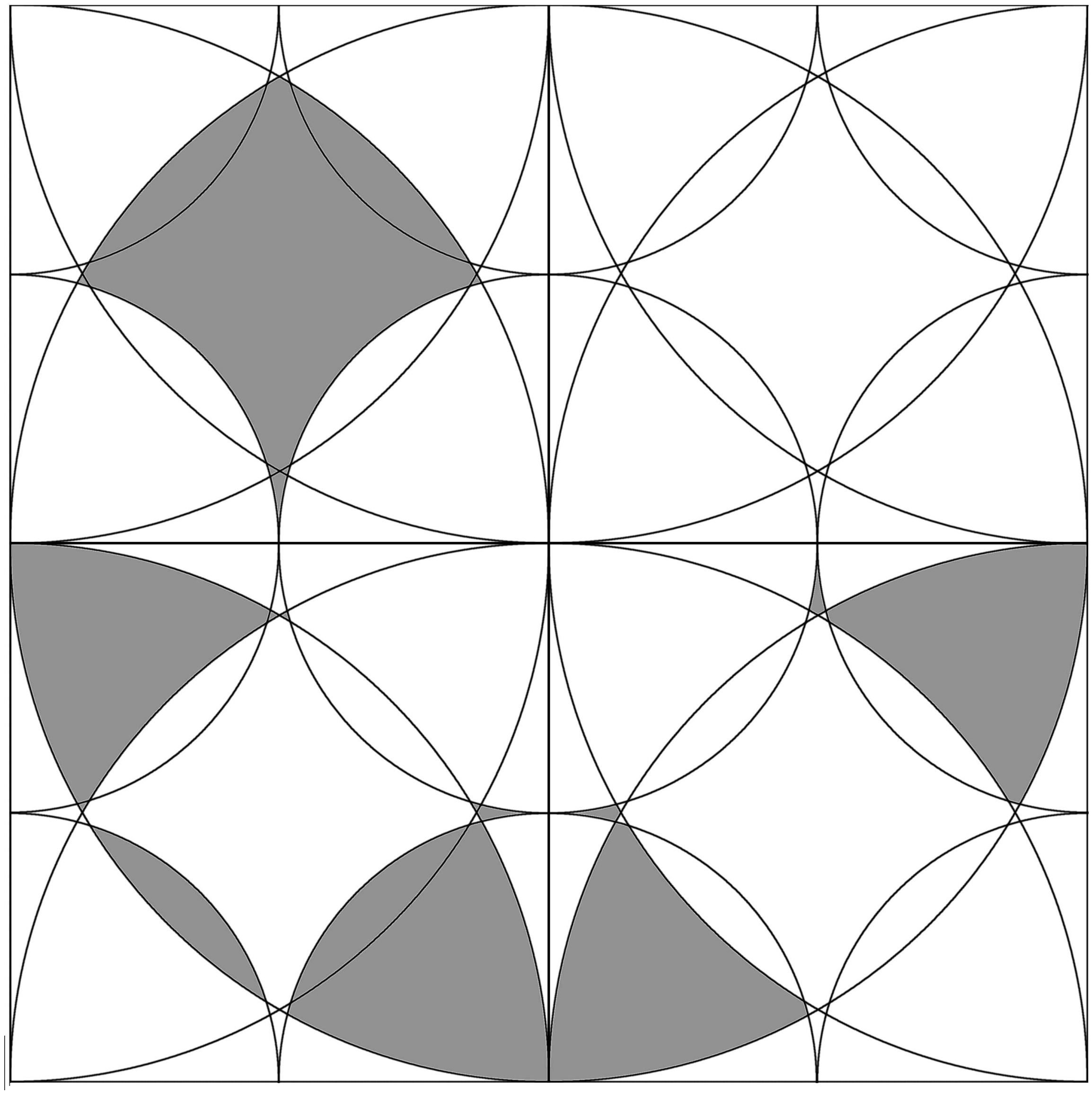

FIGURE 4

two sets will also be partitioned by its intersections with each one of $\frac{\mathbb{D}}{\sqrt{2}}$, $\frac{\mathbb{D}}{\sqrt{2}}+\mathbf{k}_1$ and the complement of the union of the latter pair of sets (see fig. 6). This, now results in a partition of Q into 17 sets.

Arguing as in the case of $A = 2I_2$, we can now obtain the sets C_1 and C_2, so that $\tilde{h}_i(\xi) = \chi_{C_i}(\xi)$, where $\xi \in \mathbb{T}^2$ and $i = 1, 2$ (see fig. 8(a) and 8(b) respectively).

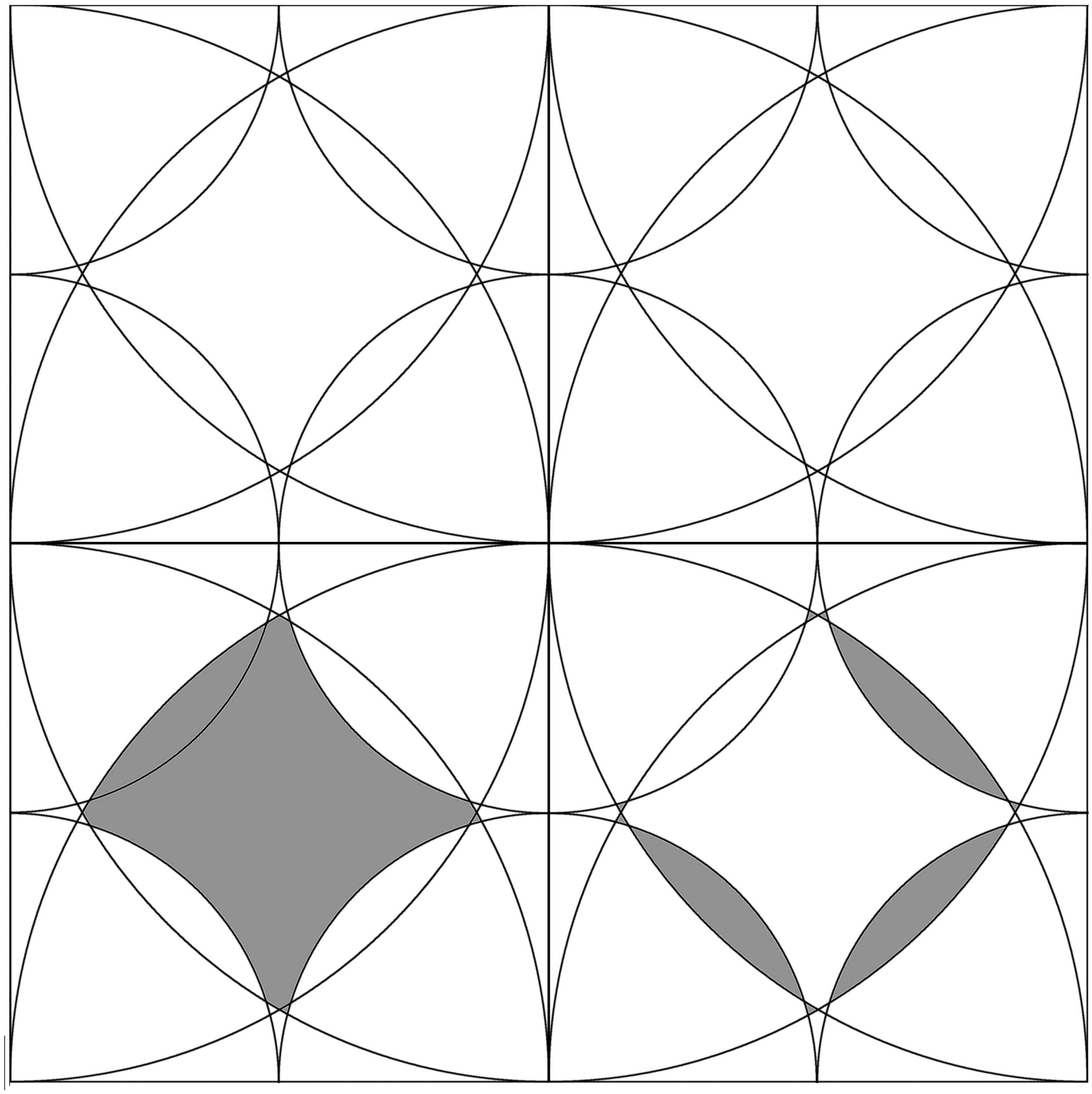

FIGURE 5

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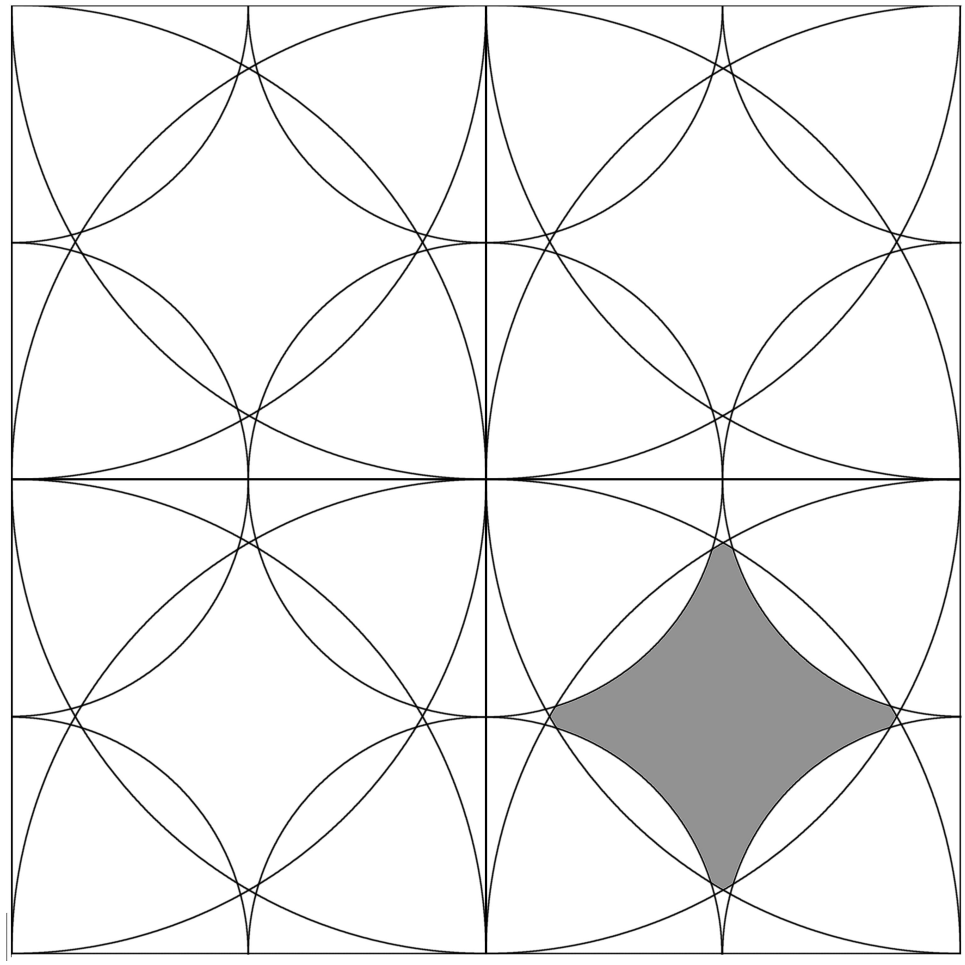

FIGURE 6

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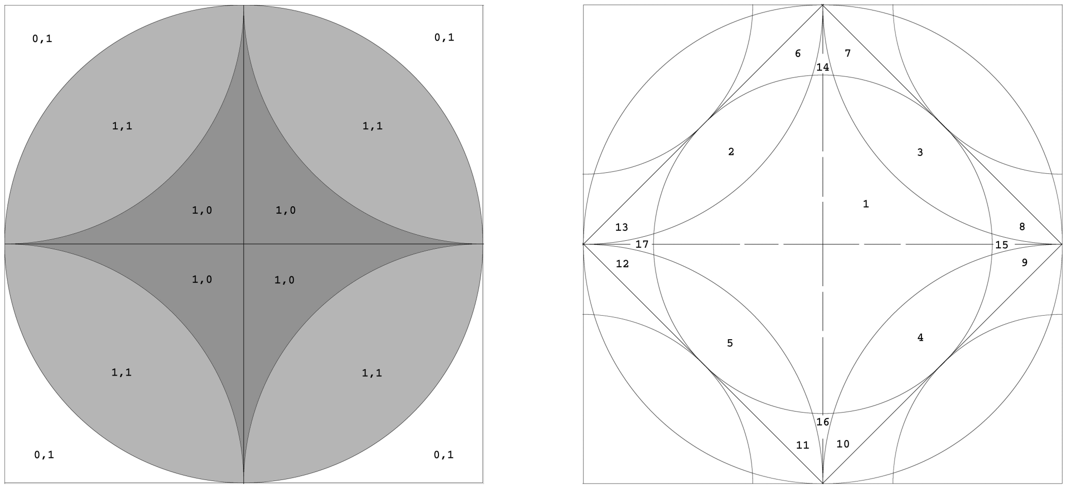
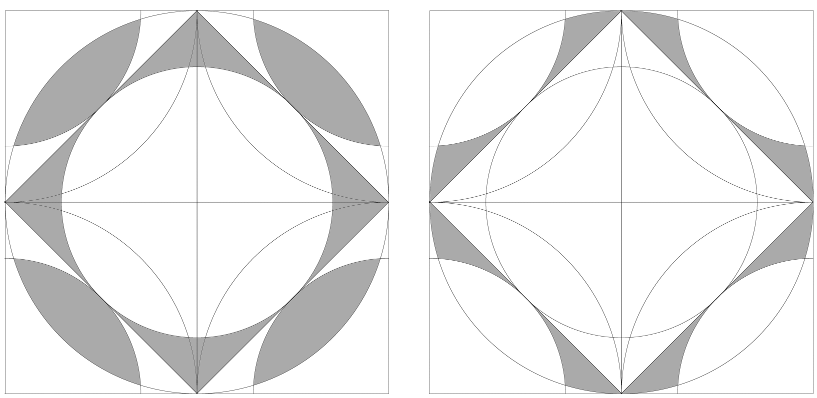


(a) (b)

FIGURE 7. (a), (b)



(a) (b)

FIGURE 8. (a), (b)

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