Central limit theorems for the shrinking target problem.

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October 24, 2013

Abstract

Suppose $B_i := B(p, r_i)$ are nested balls of radius r_i about a point p in a dynamical system (T, X, μ). The question of whether $T^i x \in B_i$ infinitely often (i.o.) for μ a.e. x is often called the shrinking target problem. In many dynamical settings it has been shown that if $E_n := \sum_{i=1}^n \mu(B_i)$ diverges then there is a quantitative rate of entry and $\lim_{n\to\infty} \frac{1}{E_n} \sum_{j=1}^n 1_{B_i}(T^i x) \to 1$ for μ a.e. $x \in X$. This is a self-norming type of strong law of large numbers. We establish self-norming central limit theorems (CLT) of the form $\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^n [1_{B_i}(T^i x) - \mu(B_i)] \to N(0,1)$ (in distribution) for a variety of hyperbolic and non-uniformly hyperbolic dynamical systems, the normalization constants are $a_n^2 \sim E[\sum_{i=1}^n 1_{B_i}(T^i x) - \mu(B_i)]^2$. Dynamical systems to

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‡MN was partially supported by NSF grant DMS 1101315.

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which our results apply include smooth expanding maps of the interval, Rychlik type maps, Gibbs-Markov maps, rational maps and, in higher dimensions, piecewise expanding maps. For such central limit theorems the main difficulty is to prove that the non-stationary variance has a limit in probability.

Contents

1 Introduction

Suppose (T, X, μ) is an ergodic dynamical system and $B_i(p)$ is a nested sequence of balls about a point $p \in X$. Recently there have been many papers concerning the behavior of the almost sure limit of the normalized sum $\frac{1}{E_n}\sum_{i=1}^{n} 1_{B_i(p)}(x)$ where $E_n := \sum_{i=1}^{n} \mu(B_i(p))$ diverges [4, 26, 11, 13, 14, 19, 24]. If the limit is known to exist almost surely then $\{B_i(p)\}$ is said to satisfy the Strong Borel Cantelli property. Many of the references we mentioned consider more general sequences of sets than nested balls. The study of hitting time statistics to a sequence of nested balls is sometimes called the shrinking target problem. In this paper we study self-norming central limit theorems for the shrinking target problem, namely the distribution limit of $\frac{1}{a_n}\sum_{i=1}^{n}[1_{B_i(p)} \circ T^i - \mu(B_i)]$ where a_n is a sequence of norming constants. One important case for applications is the case where $\mu(B_i(p)) = \frac{1}{i}$, and hence $E_n = \log n$. Our results are stated for balls satisfying $\sum_i \mu(B_i(p)) = \infty$ and $\mu(B_i(p)) \le \frac{C_2}{i^\gamma}$ where C_2 is a positive constants and $0 < \gamma \le 1$. The main difficulty is to establish that the non-stationary variance has a limit in probability. Our results are limited to non-uniformly expanding systems i.e. those without a contracting direction and are based upon the Gordin [12] martingale approximation approach (see also [27]).

More generally, this paper is also an attempt to study the statistics of non-stationary stochastic processes arising as observations (which perhaps change over time) on an underlying dynamical system (which may change over time). Conze and Raugi [6] studied similar problems for sequential expanding dynamical systems. Somewhat related results were obtained by Nándori, Szász and Varjú [30] who obtained central limit theorems in the setting in which a fixed observation $\phi : X \to \mathbb{R}$ was considered on a space on which a sequence of different transformations acted $T_i : X \to X$, preserving a common invariant measure μ. The main difficulty in [30] was also controlling the variance, but the setting in which the underlying maps change but the observation is fixed is simpler in some respects and more difficult in others.

We obtain fairly complete results in the case in which the transfer operator with respect to the invariant measure is quasicompact in the bounded variation norm. These results are contained in Proposition 5.1 and Theorem 6.4. For systems in which the transfer operator is quasicompact in a Hölder or Lipschitz space we show that under the assumption we call (SP) (derived from a Gal-Koksma lemma as formulated by Sprindzuk [38]) or a form of short returns assumption called Assumption (C) we have a central limit theorem (Theorem 3.1). Assumption (C) and the SP property have been shown to hold for generic points in a variety of non-uniformly expanding systems [5, 15, 21].

In Section 2 we discuss the set-up, describe the martingale approach we use, prove some general results on variance and discuss the SP property and Assumption (C). Section 3 gives our results under the assumption of quasi-compactness in Hölder norms and also some applications. In Section 4 we give our results when we have quasi-compactness of the transfer operator in the bounded variation norm, and we give applications to piecewise expanding maps in higher dimensions. The last section is a concluding discussion, while the Appendices describe the Gal-Koksma lemma we use and show that Assumption (C) is satisfied for generic points in many of our applications.

2 The setup.

We suppose that (T, X, μ) is an ergodic dynamical system. Let the transfer operator P be defined by $\int \phi\psi \circ T \, d\mu = \int P\phi\psi \, d\mu$ for all ϕ, $\psi \in \mathscr{L}^2(\mu)$ so that P is the adjoint of the Koopman operator $U\phi := \phi \circ T$ with respect to the invariant measure μ. Suppose $\mathcal{B}_\alpha$ is a Banach space of functions and $\|\phi\|_1 \leq C\|\phi\|_\alpha$ where $\|.\|_\alpha$ is the Banach space norm and $\|.\|_1$ is the $\mathscr{L}^1$ norm with respect to μ. We assume P restricts to an operator $P : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ such that $\|P^n\phi\|_\alpha \leq C_1\theta^n\|\phi\|_\alpha$ for all $\phi \in \mathcal{B}_\alpha$ such that $\int \phi \, d\mu = 0$. This implies exponential decay of correlations of the form, that for some $0 < \theta < 1$,

$$\left| \int \phi\psi \circ T^n \, d\mu - \int \phi \, d\mu \int \psi \, d\mu \right| \leq C\theta^n \|\phi\|_\alpha \|\psi\|_1$$

for all $\phi \in \mathcal{B}_\alpha$, $\psi \in \mathscr{L}^1(\mu)$. In our applications we will have the pairs $(BV(X), \mathscr{L}^1(\mu))$ or $(H_\gamma(X), \mathscr{L}^1(X))$ where $BV(X)$ is the space of function of bounded variation and $H_\gamma(X)$ is the space of Hölder functions of exponent γ. For example if T is a smooth uniformly expanding map of the unit interval X then $\mathcal{B}_\alpha$ could be taken as the Banach space of functions of bounded variation $BV(X)$. In this paper we will consider Lipschitz rather than Hölder functions, as our results and proofs immediately generalize to the Hölder setting with the obvious changes.

Remark 2.1 The weaker assumption of exponential decay of correlations

$$\left|\int \phi\psi\circ T^n\ d\mu - \int \phi\ d\mu \int \psi\ d\mu\right| \le C\theta^n \|\phi\|_\alpha \|\psi\|_\infty$$

implies that $\|P^n\phi\|_1 \le C\theta^n\|\phi\|_\alpha$ (by taking ψ to be $\text{sign}(P^n\phi)$) and hence P contracts exponentially in the $\mathscr{L}^1$ norm. This assumption is sufficient for all our results on variance in Section 2, with the exception of the proof of the boundedness of the terms w_j, given in Lemma 2.10 which seems to require our stronger assumption that $\|P^n\phi\|_\alpha \le C_1\theta^n\|\phi\|_\alpha$. These estimates on the growth of w_j are used in the proof of Theorem 3.1. If $\mathcal{B}_\alpha$ is the space of functions of bounded variation then the w_j terms are easily seen to be uniformly bounded under the assumption $\|P^n\phi\|_{BV} \le C\theta^n\|\phi\|_{BV}$.

Let $p \in X$ and let $B_n(p)$ be a sequence of nested balls about p such that $\mu(B_n(p)) \le \frac{C_2}{n^\gamma}$ for constants $C_2 > 0$ and $0 < \gamma \le 1$. Let $1_{B_n(p)}$ be the characteristic function of $B_n(p)$. We will sometimes write $E[\phi]$ or $\int \phi$ for the integral $\int \phi\ d\mu$ when the context is understood. Our standing assumption is that $\sum_{j=1}^n \mu(B_j(p)) \to \infty$ as $n \to \infty$.

$1_{B_n(p)}$ may not lie in $\mathcal{B}_\alpha$ but we assume we may take an approximation to it, $\tilde{\phi}_n^\alpha$ such that:

(i) $|1_{B_n(p)} - \tilde{\phi}_n^\alpha|_1 \le \frac{1}{n^3}$ and;

(ii) $\|\tilde{\phi}_n^\alpha\|_\alpha \le Cn^k$ where $C,\ k$ are independent of n;

(iii) $\tilde{\phi}_n^\alpha \ge 0$, $\tilde{\phi}_n^\alpha \ge \tilde{\phi}_{n+1}^\alpha$

Remark 2.2 If we are taking a Hölder approximation then condition (ii) is satisfied for the balls $B_i = B(p, r_i)$ if there exists $\delta(p) > 0$ and $C > 0$ such that $\mu\{x : r < d(x,p) < r + \epsilon\} < C\epsilon^{\delta(p)}$. This condition is satisfied if the invariant measure μ has a density h with respect to Lebesgue measure m such that $h \in \mathscr{L}^{1+\eta}(m)$ for some $\eta > 0$.

We define $\phi_n^\alpha = \tilde{\phi}_n^\alpha - \int \tilde{\phi}_n^\alpha$ so that $\int \phi_n^\alpha = 0$. For ease of notation we will subsequently drop the superscript α on ϕ_n^α and $\tilde{\phi}_n^\alpha$.

Define $\phi_0 = 0$ and for $n \geq 1$

$$w_n = P\phi_{n-1} + P^2\phi_{n-2} + \ldots + P^n\phi_0 = \sum_{j=1}^{n} P^j \phi_{n-j}$$

so that $w_1 = P\phi_0$, $w_2 = P\phi_1 + P^2\phi_0$, $w_3 = P\phi_2 + P^2\phi_1 + P^3\phi_0$ etc... For $n \geq 1$ define

$$\psi_n = \phi_n - w_{n+1} \circ T + w_n$$

Recall our assumptions $\|\tilde{\phi}_n\|_\alpha \leq Cn^k$(so $\|\phi_n\|_\alpha \leq \tilde{C}n^k$) and $\|P^n\phi\|_\alpha \leq C_1\theta^n\|\phi\|_\alpha$ for all $\phi \in \mathcal{B}_\alpha$ such that $\int \phi d\mu = 0$; moreover we have the monotonicity property $||\phi_{n-j}||_\alpha \leq ||\phi_n||_\alpha$, for $j < n$. These facts immediately imply that $\|w_n\|_\alpha \leq C_2\|\phi_n\|_\alpha$ (where the constant C_2 takes care of the sum of the geometric series of in θ), $\|w_n \circ T\|_\alpha \leq C_3\|\phi_n\|_\alpha$ (since $\|UP\phi\|_\alpha \leq C\|\phi\|_\alpha$ for all $\phi \in \mathcal{B}_\alpha$) and hence $\|\psi_n\|_\alpha \leq C_4\|\phi_n\|_\alpha$. Using the fact that $P(w_{n+1} \circ T) = w_{n+1}P1 = w_{n+1}$ one may show that $P\psi_n = 0$.

Since $UP(\cdot) = E[\cdot|T^{-1}\mathcal{B}]$, $P\psi_j = 0$ implies that $E[\psi_j|T^{-1}\mathcal{B}] = 0$ and in turn $E[\psi_j \circ T^j|T^{-1-j}\mathcal{B}] = 0$ (since T preserves μ). Furthermore $\psi_j \circ T^j$ is $T^{-j}\mathcal{B}$ measurable for all $j \geq 0$.

Following the approach of Gordin we will express $\sum_{j=1}^n \phi_j \circ T^j$ as the sum of a (non-stationary) martingale difference array and a controllable error term and then use the following Theorem 3.2 from Hall and Heyde [16]:

Theorem 2.3 (Theorem 3.2 [16]) *Let $\{S_{n,i}, \mathcal{F}_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be a zero-mean square-integrable martingale difference array with differences $X_{n,i}$ and let η^2 be an almost sure finite random variable. Suppose that:*

(a) $\max_i |X_{n,i}| \to 0$ *in probability;*

(b) $\sum_i X_{n,i}^2 \to \eta^2$ *in probability;*

(c) $E(\max_i X_{n,i}^2)$ *is bounded in n;*

(d) the σ-fields are nested: $\mathcal{F}_{n,i} \subset \mathcal{F}_{n+1,i}$ *for* $1 \leq i \leq k_n$, $n > 1$.

Then $S_{n,k_n} \to Z$ (in distribution) where the random variable Z has the characteristic function $E(\exp(-\frac{1}{2}\eta^2 t^2))$.

As is common in the application of martingale theory to non-invertible dynamical systems we will have to consider the natural extension so that we have a martingale in backwards time. We outline our scheme of proof.

Let (σ, Ω, m) be the natural extension of (T, X, μ). Each ψ_j lifts to to a function ψ_j^* on Ω in a natural way, $\psi_j^*(\dots\omega_{-2}\omega_{-1}.\omega_0\omega_1\dots) := \psi_j(\omega_0)$. To simplify notation we write simply ψ_j instead of ψ_j^*.

We define scaling constants by $a_n^2 = E(\sum_{j=1}^n \phi_j \circ T^j)^2$. This sequence of constants play the role of non-stationary variance. Giving estimates on the growth and non-degeneracy of a_n in this non-stationary setting is more difficult than in the usual stationary case.

We define a triangular array $X_{n,i} = \frac{1}{a_n}\psi_{n-i} \circ \sigma^{-i}$, $i = 1, \dots, n, n \in \mathbb{N}$, and put $S_{n,i} = \sum_{j=1}^i X_{n,j}$ for the partial sums (along rows). Then $X_{n,i}$ is $\mathcal{F}_i := \sigma^i \mathcal{B}_0$ measurable where $\mathcal{B}_0$ is the σ-algebra $\mathcal{B}$ lifted to Ω. Note that in Theorem 2.3 we take $\mathcal{F}_{n,i} := \mathcal{F}_i$ for all n and $k_n = n$. The $\mathcal{F}_i$ form an increasing sequence of σ-algebras. We obtain $E[S_{n,i+1}|\mathcal{F}_i] = S_{n,i} + E[X_{n,i+1}|\mathcal{F}_i]$ where by stationarity $E[X_{n,i+1}|\mathcal{F}_i] = E[\psi_{n-i-1}|\sigma\mathcal{B}_0] = 0$. Hence $E[S_{n,i+1}|\mathcal{F}_i] = S_{n,i}$ and for every $n \in \mathbb{N}$ $X_{n,i}$ is a martingale difference array with respect to $\mathcal{F}_i$.

We will then verify conditions (a), (b), (c) and (d) of Theorem 2.3. The hard part lies in establishing (b). This is in contrast with the stationary setting where condition (b) is usually a straightforward consequence of the ergodic theorem. Condition (b) is established in [30] by using [37, Lemma 3.3.], however in our setting the Lipschitz norms of the observations $\tilde{\phi}_i$ are unbounded and other techniques have to be used.

Once we have established (a), (b), (c) and (d) it follows that $\lim_{n\to\infty} \frac{1}{a_n}\sum_{j=0}^{n-1} \psi_j \circ T^j \to N(0,1)$ in distribution. In the final step we show that $\frac{1}{a_n}\sum_{j=1}^n [w_j \circ T^j - w_j \circ T^{j+1}] \to 0$ in $\mathscr{L}^1$ which implies that $\lim_{n\to\infty} \frac{1}{a_n}\sum_{j=0}^{n-1} \phi_j \circ T^j \to N(0,1)$ in distribution.

2.1 Some lemmas on variance

In this section we establish some preliminary results on the growth of the variance $E[(\sum_{j=1}^n \phi_j \circ T^j)^2]$ that will be useful in determining the scaling constants a_n.

For further reference let us notice that $\|P^n\phi\|_\alpha \le C_3\theta^n\|\phi\|_\alpha$ and $\|\phi\|_1 \le C_3\|\phi\|_\alpha$

and that there exists a constant a such that

$$\|\sum_{j>a\log i} P^j\phi_i\|_1 \leq \frac{1}{i^3}. \tag{2.1}$$

Lemma 2.4

$$\limsup_{n\to\infty}\frac{1}{E_n}E(\sum_{i=1}^n \phi_i\circ T^i)^2 \geq 1$$

where $E_n=\sum_{j=1}^n E(\phi_j^2)$.

Proof: By exponential decay of correlations and (2.1) we get for the long term interactions:

$$\sum_{j>a\log i+i}\left|\int \phi_i\circ T^i\phi_j\circ T^j\right| \leq \frac{c_1}{i^2},$$

where we used exponential decay and our bound $\|\phi_j\|_1\leq C_3\|\phi_j\|_\alpha\leq c_1 j^k$, where c_1,k are independent of j. This bound is from assumption (ii). Recall $\phi_j=\tilde{\phi}_j-\int\tilde{\phi}_j$ and $\|\tilde{\phi}_j\|_1\leq\frac{c_2}{j^\gamma}$ (for some c_2). Thus for the short term interactions we get

$$\sum_{j=i+1}^{i+a\log i}\int\phi_i\circ T^i\phi_j\circ T^j=\sum_{j=i+1}^{i+a\log i}\int\tilde{\phi}_i\circ T^i\tilde{\phi}_j\circ T^j+O(a\log iE(\tilde{\phi}_i)^2)$$

whence

$$\sum_{i=1}^n\sum_{j>i}E[\phi_i\circ T^i\phi_j\circ T^j]=\sum_{i=1}^n\sum_{j=i+1}^{i+a\log i}E[\tilde{\phi}_i\circ T^i\tilde{\phi}_j\circ T^j]+\sum_{i=1}^n O(a\log iE(\tilde{\phi}_i)^2).$$

Since

$$E(\sum_{i=1}^n\phi_i\circ T^i)^2=\sum_{i=1}^n E(\phi_i^2)+2\sum_{i=1}^n\sum_{j>i}E[\phi_i\circ T^i\phi_j\circ T^j]$$

and $\sum_{i=1}^n\sum_{j=i+1}^{i+a\log i}E[\tilde{\phi}_i\circ T^i\tilde{\phi}_j\circ T^j]+\sum_{i=1}^n a\log iE(\tilde{\phi}_i)^2\geq 0$ the lemma is proved. ∎

Lemma 2.5

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\int\phi_i\circ T^i\phi_j\circ T^j=\sum_{i=1}^n\int(\phi_i w_i)\circ T^i$$

Proof: Recalling that $\phi_0 = 0$ this follows by a direct calculation and rearrangement of terms as

$$\begin{aligned}
\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\int \phi_i\circ T^i\phi_j\circ T^j &= \sum_{j=2}^{n}\sum_{i=1}^{j-1}\int \phi_i\circ T^i\phi_j\circ T^j \\
&= \sum_{j=2}^{n}\sum_{i=1}^{j-1}\int P^{j-i}\phi_i\phi_j \\
&= \sum_{j=2}^{n}\int (\sum_{i=1}^{j-1}P^{j-i}\phi_i)\phi_j \\
&= \sum_{j=2}^{n}\int w_j\phi_j.
\end{aligned}$$

∎

The following lemma is the main result of this subsection:

Lemma 2.6

$$a_n^2 = E(\sum_{i=1}^{n}\phi_i\circ T^i)^2 = \sum_{i=1}^{n}E[\psi_i^2] - \int w_1^2 + \int w_{n+1}^2$$

Proof: Let us first observe that factoring out yields

$$\begin{aligned}
\psi_j^2 &= \phi_j^2 + 2\phi_j(w_j - w_{j+1}\circ T) + (w_j - w_{j+1}\circ T)^2 \\
&= \phi_j^2 + 2\phi_j(w_j - w_{j+1}\circ T) + w_j^2 + w_{j+1}^2\circ T - 2w_jw_{j+1}\circ T
\end{aligned}$$

which when integrated leads to

$$\begin{aligned}
\int \psi_j^2 &= \int \phi_j^2 + 2\int \phi_j(w_j - w_{j+1}\circ T) + \int w_j^2 + \int w_{j+1}^2 - 2\int w_jw_{j+1}\circ T \\
&= \int \phi_j^2 + 2\int \phi_jw_j - 2\int P\phi_jw_{j+1} + \int w_j^2 + \int w_{j+1}^2 - 2\int Pw_jw_{j+1} \\
&= \int \phi_j^2 + 2\int \phi_jw_j - 2\int P\phi_jw_{j+1} + \int w_j^2 + \int w_{j+1}^2 - 2\int (w_{j+1} - P\phi_j)w_{j+1} \\
&= \int \phi_j^2 + 2\int \phi_jw_j + \int w_j^2 - \int w_{j+1}^2.
\end{aligned}$$

Since by Lemma 2.5

$$a_n^2 = \sum_{i=1}^{n} E(\phi_i^2) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \int \phi_i \circ T^i \phi_j \circ T^j = \sum_{i=1}^{n}\left(E(\phi_i^2) + 2\int (\phi_i w_i) \circ T^i\right)$$

the statement follows by substituting $\int \psi_j^2 - \int w_j^2 + \int w_{j+1}^2$ for the terms inside the sum on the RHS and then telescoping out the expected values of w_j^2. ∎

2.2 Property (SP)

Several authors [25, 4] have used a property derived from the Gal-Koksma theorem (see Appendix) to prove the SBC property for sequences of balls. Later we will show that in certain settings the (SP) property also implies a CLT. Property SP (where SP stands for Sprindzuk Property), states that $E(S_{(m,n)})^2) \leq CE(S_{(m,n)})$ where $f_i \geq 0$ and $S_{(m,n)} = \sum_{i=m}^{n} f_i$, it is a condition that appears in one guise or another often in proofs of the Borel Cantelli property (which is easily deduced from this).

Suppose B_i are balls and let $f_i = 1_{B_i} \circ T^i$. If

$$\sum_{i=m}^{n}\sum_{j=i+1}^{n} E(f_i f_j) - E(f_i)E(f_j) \leq C\sum_{i=m}^{n} E(f_i) \qquad (SP)$$

for arbitrary integers $n > m$ then the balls are said to have the (SP) property.

2.3 Short returns and Assumption (C)

In this section we discuss a condition on short return times first considered, to our knowledge, by P. Collet [5]. We have called it Assumption (C), after Collet. This condition has been used to establish extreme value statistics [5, 21, 15] and dynamical Borel-Cantelli lemmas [14, 19]. Assumption (C) is a strong control on measure of points making short returns. If Assumption (C) holds for a point p then the measure of a set of points in nested balls $B_i(p)$ about p returning to $B_i(p)$ in a time interval smaller than an integer power of $-\log \mu(B_i(p))$ is smaller than $\mu(B_i(p))^\eta$ where η is greater than one. Note by Kac's theorem the expected return time to $\mu(B_i(p))$ is of order $\mu(B_i(p))^{-1}$, so an integer power of

$-\log\mu(B_i(p))$ is indeed a very short return. This condition fails for periodic points, as a fixed fraction of the mass of a ball returns after the period. Heuristically if the first return times to $B_i(p)$ follow an exponential law (which one somehow expects for generic points) then $\lim_{i\to\infty}\frac{1}{\mu B_i(p))}\mu\{x\in B_i(p):\tau(x)>\frac{t}{\mu(B_i(p))}\}\to e^{-t}$ and hence $\lim_{i\to\infty}\frac{1}{\mu B_i(p))}\mu\{x\in B_i(p):\tau(x)\le\frac{t}{\mu(B_i(p))}\}\to 1-e^{-t}\sim t$ (for small t). Suppose now we could solve for $\frac{t}{\mu(B_i(p))}=(-\log(\mu(B_i(p)))^k$, we would then have $\mu\{x\in B_i(p):\tau(x)\le(-\log(\mu(B_i(p)))^k\}\sim(-\log(\mu(B_i(p)))^k\mu(B_i(p))^2$. Note that our assumption $\mu(B_i)\le\frac{C_2}{i^\gamma}$ implies that $(-\log\mu(B_i))^k\ge C(\log(i))^k$ for large i.This train of thought makes Assumption (C) seem reasonable for generic points.

Suppose therefore that $p\in X$ and $B_i(p)$ is a nested sequence of balls centered at a point p, with $\lim_i\mu(B_i(p))=0$.

Assumption (C): We say $(B_i(p))$ satisfies Assumption (C) if there exists $\eta(p)\in(0,1)$ and $\kappa(p)>1$ such that for all i sufficiently large

$$\mu(B_i(p)\cap T^{-r}B_i(p))\le\mu(B_i(p))^{1+\eta}$$

for all $r=1,\dots,(\log i)^\kappa$.

If $(B_i(p))$ satisfies Assumption (C) then we can say more about the behavior of the constants a_n.

Remark 2.7 Note that our assumption $\mu(B_i)\le\frac{C_2}{i^\gamma}$ implies that $(-\log\mu(B_i))^k\ge C(\log(i))^k$ for large i.

Lemma 2.8 *Under Assumption (C) there exists a constant C_4 and some large a so that*

$$\int|\phi_j w_j|\le C_4\mu(B_{j-a\log j})^{1+\eta}\log j.$$

Proof: By the contraction property of the transfer operator one has as in (2.1) for a sufficiently large constant a

$$\sum_{i<j-a\log j}\int\phi_j P^{j-i}\phi_i\le\mu(B_{j-a\log j})^2.$$

Let $\phi_j = \tilde\phi_j - \int \tilde\phi_j$ where $\tilde\phi_j$ is the $\mathcal{B}_\alpha$ approximation to $1_{B_j(p)}$. Hence we obtain in the $\mathscr{L}^1$-norm: (as $\tilde\phi_j \geq 0$)

$$\begin{aligned}
\int |\phi_j w_j| &\leq \sum_{n=1}^{a\log j}\left(\int \tilde\phi_j P^n \tilde\phi_{j-n} + \int \tilde\phi_{j-n}\int\tilde\phi_j + \int\tilde\phi_j \int P^n\tilde\phi_{j-n} + \int\tilde\phi_j\int\tilde\phi_{j-n}\right)\\
&\quad +\mathcal{O}\left((j-a\log j)\mu(B_{j-a\log j})^2\right)\\
&= \sum_{n=1}^{a\log j}\left(\int \tilde\phi_j P^n\tilde\phi_{j-n} + 3\mu(\tilde\phi_{j-n})\mu(\tilde\phi_j)\right) + \mathcal{O}(j)\mu(B_{j-a\log j})^2\\
&= \sum_{n=1}^{a\log j}\int \tilde\phi_j P^n\tilde\phi_{j-n} + \mathcal{O}(j)\mu(B_{j-a\log j})^2,
\end{aligned}$$

Now by Assumption (C) we have

$$\int \tilde\phi_j P^n\tilde\phi_{j-n} \leq \int \tilde\phi_{j-n}\circ T^n\tilde\phi_{j-n} \leq \mu(B_{j-n}\cap T^{-n}B_{j-n}) \leq c_1\mu(B_{j-n})^{1+\eta},$$

for $n \leq a\log j$, and thus

$$\sum_{n=1}^{a\log j}\int \tilde\phi_j P^n\tilde\phi_{j-n} \leq c_2 a\mu(B_{j-a\log j})^{1+\eta}\log j,$$

proving the lemma. ∎

Lemma 2.9 *If $(B_i(p))$ satisfies Assumption (C) then*

$$\lim_{n\to\infty}(E(\sum_{i=1}^n \phi_i\circ T^i)^2)/(\sum_{i=1}^n E[\phi_i^2]) = 1$$

Proof: Rearranging the sums yields by Lemma 2.5

$$\begin{aligned}
E(\sum_{i=1}^n \phi_i\circ T^i)^2 &= \sum_{i=1}^n E[\phi_i^2] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n\int \phi_i\circ T^i\phi_j\circ T^j\\
&= \sum_{i=1}^n E[\phi_i^2] + 2\sum_{j=2}^n\int w_j\phi_j
\end{aligned}$$

and hence the result follows by Lemma 2.8 as $\eta > 1$. ∎

2.4 Bounds on w_j

We now assume that $\|\phi\|_\infty \le C\|\phi\|_\alpha$ which under our assumption on the transfer operator implies that for a mean-zero function $\phi \in \mathcal{B}$, $\|P^n\phi\|_\infty \le C\theta^n\|\phi\|_\alpha$ for some C, $0<\theta<1$ independently of ϕ. For example if $\|.\|_\alpha$ were the Banach space of Hölder functions of exponent α on the unit interval then $\|\phi\|_\infty \le C\|\phi\|_\alpha$. In the BV or quasi-Hölder norm indicator functions are bounded, and the proof that w_j is uniformly bounded is straightforward in this case; we would like to stress that the next result is obtained under the general assumption that $\mu(B_n(p)) \le \frac{C_2}{n^\gamma}$.

Lemma 2.10 *Assume $\|P^n\phi\|_\infty \le C\theta^n\|\phi\|_\alpha$ then there exists a constant C_5 such that $\|w_j\|_\infty < C_5$ for all j.*

Proof: For some $a>0$ we can achieve $\sum_{j=\lfloor a\log n\rfloor}^{n} |P^j\phi_{n-j}|_\infty \le c_1\sum_{j=\lfloor a\log n\rfloor}^{n}\theta^j(n-j)^k = \mathcal{O}(n^{-2})$ and in particular $|P^j\phi_{n-j}|_\infty = \mathcal{O}(n^{-2})$ for all $j \ge \lfloor a\log n\rfloor$ and all n. As in the previous lemma let $\tilde\phi_j$ be the $\mathcal{B}_\alpha$ approximation for 1_{B_j} and $\phi_j = \tilde\phi_j - \mu(\tilde\phi_j)$. In view of the tail estimate it is only necessary to bound $\sum_{j=1}^{\lfloor a\log n\rfloor} P^j\phi_{n-j}$ independently of n.

(i) Bound from below: Since $\phi_j \ge -\mu(\tilde\phi_j) \ge -c_2\mu(B_j) \ge -\frac{c_3}{j^\gamma}$ $(c_2,c_3>0)$ one obtains $\sum_{j=1}^{\lfloor a\log n\rfloor} P^j\phi_{n-j} \ge \sum_{j=1}^{\lfloor a\log n\rfloor}\frac{c_3}{(n-j)^\gamma} \ge \frac{-c_4\log n}{n^\gamma}$ for some constant c_4 independent of j and n. Hence $w_n \ge -c_5$ for some $c_5>0$ and all n.

(ii) Bound from above: Since $1_{B_{j+1}} \le 1_{B_j}$ one has $\tilde\phi_{j+1} \le \tilde\phi_j$ and in particular $\mu(\tilde\phi_{j+1}) \le \mu(\tilde\phi_j)$. Hence $\phi_{j+1}-\phi_j \le \mu(\tilde\phi_j)-\mu(\tilde\phi_{j+1})$ and (as $\phi_0 = 0$)

$$
\begin{aligned}
w_m - w_{m-1} &= \sum_{j=1}^{m-1} P^j(\phi_{m-j}-\phi_{m-1-j}) + P^m\phi_0 \\
&\le \sum_{j=1}^{m-1}\Big(\mu(\tilde\phi_{m-1-j})-\mu(\tilde\phi_{m-j})\Big) \\
&\le \sum_{j=1}^{\lfloor a\log m\rfloor}\Big(\mu(\tilde\phi_{m-1-j})-\mu(\tilde\phi_{m-j})\Big) + \mathcal{O}(m^{-2}).
\end{aligned}
$$

Consequently ($w_1 = P\phi_0 = 0$)

$$\begin{aligned}
w_n &= \sum_{m=2}^{n}(w_m - w_{m-1}) + w_1 \\
&\leq \sum_{m=2}^{n}\left(\sum_{j=1}^{\lfloor a\log m\rfloor}\Big(\mu(\tilde{\phi}_{m-1-j}) - \mu(\tilde{\phi}_{m-j})\Big) + \mathcal{O}(m^{-2})\right) \\
&= \sum_{j=1}^{\lfloor a\log n\rfloor}\ \sum_{m=2\vee\lceil e^{\frac{j}{a}}\rceil}^{n}\Big(\mu(\tilde{\phi}_{m-1-j}) - \mu(\tilde{\phi}_{m-j}) + \mathcal{O}(m^{-2})\Big) \\
&= \sum_{j=1}^{\lfloor a\log n\rfloor}\Big(\mu(\tilde{\phi}_{2\vee\lceil e^{\frac{j}{a}}\rceil-j}) - \mu(\tilde{\phi}_{n-j}) + \mathcal{O}((2\vee e^{\frac{j}{a}})^{-1})\Big) \\
&\leq c_6
\end{aligned}$$

for a constant c_6 independent of n because

$$\sum_{j=1}^{\lfloor a\log n\rfloor}\mu(\tilde{\phi}_{n-j}) \leq c_7\frac{a\log n}{n^{\gamma}} \to 0$$

as $n\to\infty$ and

$$\sum_{j=1}^{\lfloor a\log n\rfloor}\mu(\tilde{\phi}_{2\vee\lceil e^{\frac{j}{a}}\rceil-j}) \leq c_8\sum_{j=1}^{\lfloor a\log n\rfloor}(e^{\frac{j}{a}})^{-\gamma} = \mathcal{O}(1)$$

for constants c_7, c_8 independent of n. ∎

3 Decay in Lipschitz versus $\mathscr{L}^1$

We take $\mathcal{B}_\alpha$ to be the space of Lipschitz functions, the arguments we give hold for Hölder norms with obvious modification. We assume that the transfer operator P, when restricted to $Lip(X)$, contracts exponentially:

$$||P^n\phi||_{Lip} \leq C\theta^n||\phi||_{Lip} \qquad (3.1)$$

for all Lipschitz functions ϕ such that $\int\phi\,d\mu = 0$ where $\theta\in(0,1)$ and C are independent of ϕ.

This implies

$$\left| \int \phi\psi \circ T^n \, d\mu - E[\phi]E[\psi] \right| \le C\theta^n \|\phi\|_{Lip} \, \|\psi\|_{\mathscr{L}^1} \tag{3.2}$$

for the same $\theta \in (0,1)$ and C independent of ϕ, ψ.

For a sequence of (nested) balls B_i we put $E_n = \sum_{i=1}^n \mu(B_i)$ and $S_n = \sum_{i=1}^n 1_{B_i} \circ T^i$ for the 'hit counter' for an orbit segment of length n. The sequence of balls B_i satisfies the *strong Borel-Cantelli* (SBC) property if

$$\lim_{n\to\infty} \frac{S_n(x)}{E_n} = 1 \tag{3.3}$$

for almost every $x \in X$.

Theorem 3.1 *Assume that the transfer operator, when restricted to Lip(X), contracts exponentially as in (3.1) for some $\theta \in (0,1)$.*

Suppose $B_i(p)$ be nested balls about a point p such that $\sum_i \mu(B_i) = \infty$ and $\mu(B_i(p)) \le \frac{C_2}{i^\gamma}$ for constants $C_2 > 0$ and $0 < \gamma \le 1$. Let $a_n^2 = E(\sum_{i=1}^n (1_{B_i} \circ T^i - \mu(B_i)))^2$.

(I) If the nested sequence of balls $(B_i(p))$ satisfies Assumption (C) and the SBC property (3.3) then

$$\frac{1}{a_n} \sum_{j=1}^n \big(1_{B_i} \circ T^i - \mu(B_i)\big) \to N(0,1)$$

in distribution.

(II) If $(B_i(p))$ has the SP property then

$$\frac{1}{a_n} \sum_{i=1}^n (1_{B_i} \circ T^i - \mu(B_i)) \to N(0,1).$$

Proof: We will let $\phi_j = \tilde{\phi}_j - \int \tilde{\phi}_j$,where $\tilde{\phi}_j$ be a Lipschitz approximation to 1_{B_j}, such that

$$\begin{cases} \|\tilde{\phi}_j - 1_{B_j}\|_1 & < & \frac{1}{j^3} \\ \|\tilde{\phi}_j\|_{Lip} & \le & Cj^k \\ \tilde{\phi}_j & \ge 0 \end{cases}.$$

We define $w_n = P\phi_{n-1} + P^2\phi_{n-2} + \ldots + P^n\phi_0$ and put $\psi_n = \phi_n - w_{n+1} \circ T + w_n$. Then $P\psi_n = P\phi_n - w_{n+1} + \sum_{j=2}^n P^j\phi_{n-j+1} = 0$ which corresponds to $\int \psi_n \chi \circ T\, d\mu = \int \chi P\psi_n\, d\mu = 0$ for any integrable χ. Note that $||\phi_j||_\infty \leq ||\phi_j||_{Lip}, ||\phi_j||_1 \leq ||\phi_j||_{Lip}$.

Lemma 3.2 *There exist constants C_6, k so that*
(I) $\|w_n\|_{Lip} \leq C_6 n^k$,
(II) $\|w_n\|_\infty \leq C_6$,
(III) $\|w_n\|_1 \leq C_6 \frac{\log n}{n^\gamma}$.

Proof of Lemma 3.2. (I) By the contraction of the transfer operator for Lipschitz continuous functions one obtains

$$\|w_n\|_{Lip} \leq \sum_{j=1}^n \|P^j\phi_{n-j}\|_{Lip} \leq \sum_{j=1}^n C_1\theta^j \|\phi_{n-j}\|_{Lip} \leq c_1 n^k$$

(II) Is a consequence of Lemma 2.10.

(III) For sufficiently large a we get

$$\begin{aligned}
||w_n||_1 &\leq \sum_{j=1}^{a\log n} ||P^j\phi_{n-j}||_1 + \sum_{j=a\log n+1}^{n} ||P^j\phi_{n-j}||_1 \\
&\leq \sum_{j=1}^{a\log n} ||\phi_{n-j}||_1 + \sum_{j=a\log n+1}^{n} ||P^j\phi_{n-j}||_{Lip} \\
&\leq \sum_{j=1}^{a\log n} ||\phi_{n-j}||_1 + \sum_{j=a\log n+1}^{n} C_1\theta^j||\phi_{n-j}||_{Lip} \\
&\leq C(a\log n)\mu(B_{n-a\log n}) + c_4\frac{\log^2 n}{n^2} \\
&\leq c_5\frac{\log n}{n^\gamma}
\end{aligned}$$

for some c_4, c_5 independent of n.

Now put $C_6 = \max(c_1, c_5)$. ∎

As before let (σ, Ω, m) be the natural extension of (T, X, μ) and put $a_n^2 = E(\sum_{j=1}^n \phi_j \circ T^j)^2$ for the rescaling factors where the ψ_j lift to Ω in a natural way. Again we put $X_{n,i} = \frac{1}{a_n}\psi_{n-i} \circ \sigma^{-i}$, $i = 1, \ldots, n$ which are $\mathcal{F}_i = \sigma^i\mathcal{B}_0$ measurable where $\mathcal{B}_0$ is the

σ-algebra $\mathcal{B}$ lifted to Ω. The $\mathcal{F}_i$ form an increasing sequence of σ-algebras. We put $S_{n,i} = \sum_{j=1}^{i} X_{n,j}$, $i = 1, \ldots, n$ ($k_n = n$), and obtain $E[S_{n,i+1}|\mathcal{F}_i] = S_{n,i} + E[X_{n,i+1}|\mathcal{F}_i]$ but by stationarity $E[X_{n,i+1}|\mathcal{F}_i] = E[\phi_{n-i-1}|\sigma\mathcal{B}] = 0$. Hence $E[S_{n,i+1}|\mathcal{F}_i] = S_{n,i}$ and $X_{n,i}$ is a martingale difference array with respect to $\mathcal{F}_i$.

Condition (d) clearly holds; instead conditions (a) and (c) simply follow since $||\phi_n||_\infty$ is bounded and a_n tends to infinity.

We now prove (I) and show that under Assumption (C), $\sum_{i=1}^{n} X_{n,i}^2 \to 1$ in probability and hence condition (b) holds.

Lemma 3.3

$$\frac{1}{a_n^2} \sum_{j=1}^{n} \psi_j^2 \circ T^j \to 1$$

in probability as $n \to \infty$.

Proof. We follow an argument given by Peligrad [31]. As $\psi_j = \phi_j + w_j - w_{j+1} \circ T$ we obtain

$$\begin{aligned}
\psi_j^2 &= \phi_j^2 + 2\phi_j w_j + w_j^2 + w_{j+1}^2 \circ T - 2w_{j+1} \circ T(\phi_j + w_j) \\
&= \phi_j^2 + 2\phi_j w_j + w_j^2 + w_{j+1}^2 \circ T - 2w_{j+1} \circ T(\psi_j + w_{j+1} \circ T) \\
&= \phi_j^2 + (w_j^2 - w_{j+1}^2 \circ T) - 2\psi_j w_{j+1} \circ T + 2\phi_j w_j.
\end{aligned}$$

We want to sum over $j = 1, \ldots, n$ and normalize by $\log n$ and wish to estimate the error terms which are the last four terms on the RHS. The terms $w_j^2 - w_{j+1}^2 \circ T$ are bounded and telescope so may be neglected.

In order to estimate the third of the error terms, $\psi_j w_{j+1} \circ T$ we proceed like Peligrad (page 9) using a truncation argument. Let $w_j^\epsilon = w_j 1_{\{|w_j| \leq \epsilon a_n\}}$, where for simplicity of notation we have left out the dependence on n. Then

$$\int \left(\sum_{j=1}^{n} \psi_j \circ T^j w_{j+1}^\epsilon \circ T^{j+1} \right)^2 = \sum_{j=1}^{n} \int \left(\psi_j \circ T^j w_{j+1}^\epsilon \circ T^{j+1} \right)^2 \leq \epsilon^2 a_n^2 \sum_{j=1}^{n} \int \psi_j^2$$

since the cross terms vanish (for $j > i$), as

$$\begin{aligned}\int(\psi_j w^\epsilon_{j+1}\circ T)\circ T^j(\psi_i w^\epsilon_{i+1}\circ T)\circ T^i &= \int(\psi_j w^\epsilon_{j+1}\circ T)\circ T^{j-i}(\psi_i w^\epsilon_{i+1}\circ T)\\ &= \int(\psi_j w^\epsilon_{j+1}\circ T)\circ T^{j-i-1}P(\psi_i w^\epsilon_{i+1}\circ T)\\ &= \int(\psi_j w^\epsilon_{j+1}\circ T)\circ T^{j-i-1}w^\epsilon_{i+1}P\psi_i = 0\end{aligned}$$

as $P(\psi_i w^\epsilon_{i+1}\circ T) = w^\epsilon_{i+1}P\psi_i$.

For any $a > \epsilon$ we obtain using Tchebycheff's inequality (on the second term):

$$\begin{aligned}&\mu\left(\left|\frac{1}{a_n^2}\sum_{j=1}^n \psi_j\circ T^j w_{j+1}\circ T^{j+1}\right| > a\right)\\ \le\ &\mu\left(\max_{1\le j\le n}\left|w_{j+1}\circ T^{j+1}\right| > \epsilon a_n\right) + \mu\left(\left|\frac{1}{a_n^2}\sum_{j=1}^n \psi_j\circ T^j w^\epsilon_{j+1}\circ T^{j+1}\right| > a\right)\\ \le\ &\mu(\max_{1\le j\le n}|w_{j+1}\circ T^{j+1}| > \epsilon a_n) + \frac{\epsilon^2}{a^2a_n^2}\sum_{j=1}^n\int\psi_j^2\\ =\ &\mu(\max_{1\le j\le n}|w_j\circ T^{j+1}| > \epsilon a_n) + c_1\frac{\epsilon^2}{a^2}.\end{aligned}$$

In the last line we used $\sum_{j=1}^n E[\psi_j^2]\sim a_n$ by Lemmata 2.6 and 2.8. By boundedness of the w_j (Lemma 2.10) one gets that $P(\max_{1\le j\le n}|w_{j+1}\circ T^{j+1}| > \epsilon a_n)\to 0$ for every $\epsilon > 0$ as $n\to\infty$. Choosing $a = \epsilon^{\frac12}$ we conclude that $\frac{1}{a_n^2}\sum_{j=1}^n\psi_j\circ T^j w_{j+1}\circ T^{j+1}$ converges to zero in probability as $n\to\infty$.

For the fourth error term $\frac{1}{a_n^2}2\sum_{j=1}^n(\phi_j w_j)\circ T^j$ we obtain by Lemma 2.9:

$$\left\|\sum_{j=1}^n(\phi_j w_j)\circ T^j\right\|_1 \le \sum_{j=1}^n\|\phi_j w_j\|_1 \le c_2\sum_{j=1}^\infty \mu(B_{j-a\log j})^{1+\eta}\log j$$

Thus $\frac{2}{a_n^2}\sum_{j=1}^n(\phi_j w_j)\circ T^j\to 0$ in probability.

Since the term $\frac{1}{a_n^2}\sum_{j=1}^n\phi_j^2\circ T^j$ converges to 1 almost surely by the SBC property the proof is complete. ∎

Lemma 3.3 completes the proof of part (I) of the theorem. In order to show (II) we proceed as in the proof of (I) except for the verification of condition (b). We will prove an SBC property for $\phi_j^2 + 2w_j\phi_j$. Decomposing $\phi_j = \tilde{\phi}_j - \mu(\tilde{\phi}_j)$ and defining $\tilde{w}_j = P\tilde{\phi}_{j-1} + ... + P^{[a\log j]}\tilde{\phi}_{j-[a\log j]}$ we see that $\|\phi_j^2 - \tilde{\phi}_j^2\|_1 \leq C\mu(B_j)^2$ and $\|w_j\phi_j - \tilde{w}_j\tilde{\phi}_j\|_1 \leq C\mu(B_{j-a\log j})^2 \log j$. Note that both $\tilde{w}_j$ and $\tilde{\phi}_j$ are positive functions and moreover that similarly to Lemma 3.2 there exists a constant $\tilde{C}_6$ such that $\|\tilde{w}_j\|_\infty \leq \tilde{C}_6$ and $\|\tilde{w}_j\|_{Lip} \leq \tilde{C}_6 j^k$. Let $\mathcal{E}_n := \sum_{j=1}^n E[\tilde{\phi}_j^2 + 2\tilde{w}_j\tilde{\phi}_j]$. It suffices to consider the sequence $\tilde{\phi}_j^2 + 2\tilde{w}_j\tilde{\phi}_j$. This is because $\sum_{j=1}^n E[\tilde{\phi}_j^2 + 2\tilde{w}_j\tilde{\phi}_j] = \sum_{j=1}^n E[\phi_j^2 + 2w_j\phi_j] + \sum_{j=1}^n (C\mu(B_j)^2 + C\mu(B_{j-a\log j})^2 \log j)$ and hence μ almost surely,

$$\frac{1}{\mathcal{E}_n}\sum_{j=1}^n (\tilde{\phi}_j^2 \circ T^j + 2(\tilde{w}_j\tilde{\phi}_j)\circ T^j) = \frac{1}{\mathcal{E}_n}\sum_{j=1}^n (\phi_j^2 \circ T^j + 2(w_j\phi_j)\circ T^j).$$

We will use Proposition 8.1, a form of the Gal and Koksma theorem as stated by Sprindzuk to show that $\frac{1}{\mathcal{E}_n}\sum_{j=1}^n \tilde{\phi}_j^2 \circ T^j + 2(\tilde{w}_j\tilde{\phi}_j)\circ T^j \to 1$ almost surely. For this we want to use Proposition 8.1 with $f_j = \tilde{\phi}_j^2 + 2\tilde{w}_j\tilde{\phi}_j$, $g_j = \int f_j$ and h_j to be determined below. We need to estimate the terms in $\int \left(\sum_{i=m}^n (\tilde{\phi}_j^2 + 2\tilde{w}_j\tilde{\phi}_j - \int \tilde{\phi}_j^2 + 2\tilde{w}_j\tilde{\phi}_j)\right)^2$.

In order to verify the condition of the proposition we look at the three individual sums of terms $\int \tilde{\phi}_j \circ T^{j-i}\tilde{\phi}_i$, $\int (\tilde{\phi}_j\tilde{w}_j)\circ T^{j-i}\tilde{\phi}_i\tilde{w}_i$ and $\int (\tilde{\phi}_j\tilde{w}_j)\circ T^{j-i}\tilde{\phi}_i$ as follows:
(i) The fact that condition (SP) holds for the functions $\tilde{\phi}_j$ implies

$$\sum_{i=m}^n \sum_{j=i+1}^n \int \tilde{\phi}_j \circ T^{j-i}(\tilde{\phi}_i) - E[\tilde{\phi}_j]E[\tilde{\phi}_i] \leq c_1 \sum_{i=m}^n E[\tilde{\phi}_j]. \tag{3.4}$$

Since $E(\tilde{\phi}_j^2) - E(\tilde{\phi}_j) = \mathcal{O}(j^{-k})$ we obtain

$$\sum_{i=m}^n \sum_{j=i+1}^n \int \tilde{\phi}_j^2 \circ T^{j-i}\tilde{\phi}_i^2 - E[\tilde{\phi}_j^2]E[\tilde{\phi}_i^2] \leq c_1 \sum_{i=m}^n E[\tilde{\phi}_j] + \sum_{i=m}^n \mathcal{O}(i^{-k+1}).$$

(ii) Here we estimate the sums of the terms $\int (\tilde{\phi}_j\tilde{w}_j)\circ T^{j-i}(\tilde{\phi}_i\tilde{w}_i) - E[\tilde{\phi}_j\tilde{w}_j]E[\tilde{\phi}_i\tilde{w}_i]$. By exponential decay of correlations one has for some constant $\beta > 0$

$$\sum_{j=i+1}^n \left(\int \tilde{\phi}_j \circ T^{j-i}\tilde{\phi}_i - E[\tilde{\phi}_j]E[\tilde{\phi}_i]\right) = \sum_{j=i+1}^{i+\beta\log i} \left(\int \tilde{\phi}_j \circ T^{j-i}\tilde{\phi}_i - E[\tilde{\phi}_j]E[\tilde{\phi}_i]\right) + \mathcal{O}(i^{-k})$$

and therefore

$$\sum_{j=i+1}^{i+\beta\log i} \int \tilde{\phi}_j \circ T^{j-i}\tilde{\phi}_i \leq \sum_{j=i+1}^{i+\beta\log i} E[\tilde{\phi}_j]E[\tilde{\phi}_i] + \mathcal{O}(i^{-k}) + \sum_{j=i+1}^n \left(\int \tilde{\phi}_j \circ T^{j-i}(\tilde{\phi}_i) - E[\tilde{\phi}_j]E[\tilde{\phi}_i]\right)$$

Using the uniform bound on $|\tilde{w}_j|_\infty$ this implies

$$\sum_{j=i+1}^{i+\beta\log i}\int(\tilde{\phi}_j\tilde{w}_j)\circ T^{j-i}(\tilde{\phi}_i\tilde{w}_i)$$
$$\leq \tilde{C}_6^2\left(\sum_{j=i+1}^{i+\beta\log i}E[\tilde{\phi}_j]E[\tilde{\phi}_i]+\mathcal{O}(i^{-k})+\sum_{j=i+1}^{n}\left(\int\tilde{\phi}_j\circ T^{j-i}(\tilde{\phi}_i)-E[\tilde{\phi}_j]E[\tilde{\phi}_i]\right)\right)$$

By decay of correlation and since $\|\tilde{w}_j\tilde{\phi}_j\|_\alpha\leq\tilde{C}_6 j^k$ we get with a possibly larger β that

$$\sum_{j=i+1}^{n}\int(\tilde{\phi}_j\tilde{w}_j)\circ T^{j-i}(\tilde{\phi}_i\tilde{w}_i)-E[\tilde{\phi}_j\tilde{w}_j]E[\tilde{\phi}_i\tilde{w}_i]$$
$$=\sum_{j=i+1}^{i+\beta\log i}\left(\int(\tilde{\phi}_j\tilde{w}_j)\circ T^{j-i}(\tilde{\phi}_i\tilde{w}_i)-E[\tilde{\phi}_j\tilde{w}_j)]E[\tilde{\phi}_i\tilde{w}_i]\right)+\mathcal{O}(i^{-k})$$
$$\leq \tilde{C}_6^2\sum_{j=i+1}^{i+\beta\log i}E[\tilde{\phi}_j]E[\tilde{\phi}_i]+\tilde{C}_6^2\sum_{j=i+1}^{n}\left(\int\tilde{\phi}_j\circ T^{j-i}(\tilde{\phi}_i)-E[\tilde{\phi}_j]E[\tilde{\phi}_i]\right)+\mathcal{O}(i^{-k})$$
$$\leq \tilde{C}_6^2\left(E[\tilde{\phi}_i]^2\beta\log i+\sum_{j=i+1}^{n}\left(\int\tilde{\phi}_j\circ T^{j-i}(\tilde{\phi}_i)-E[\tilde{\phi}_j]E[\tilde{\phi}_i]\right)\right)+\mathcal{O}(i^{-\tilde{k}})$$

for some $\tilde{k}>1$. Since $E[\tilde{\phi}_i]^2\log i\leq c_2E[\tilde{\phi}_i]$ $\forall i$ and some c_2 and since for every m,n inequality (3.4) holds we now obtain

$$\sum_{i=m}^{n}\sum_{j=i+1}^{n}\left(\int(\tilde{\phi}_j\tilde{w}_j)\circ T^{j-i}(\tilde{\phi}_i\tilde{w}_i)-E[\tilde{\phi}_j\tilde{w}_j]E[\tilde{\phi}_i\tilde{w}_i]\right)\leq c_3\sum_{i=m}^{n}E[\tilde{\phi}_i]+\mathcal{O}(i^{-\tilde{k}})$$

(iii) A similar argument shows that for the 'mixed' terms

$$\sum_{i=m}^{n}\sum_{j=i+1}^{n}\int(\tilde{\phi}_j\tilde{w}_j)\circ T^{j-i}\tilde{\phi}_i^2-E[\tilde{\phi}_j\tilde{w}_j]E[\tilde{\phi}_i^2]\leq c_4\sum_{i=m}^{n}E[\tilde{\phi}_j]+\sum_{i=m}^{n}\left(\mathcal{O}(i^{-\tilde{k}})\right).$$

Combining (i), (ii) and (iii) yields for all $m<n$ and some constant c_5:

$$\int\left(\sum_{i=m}^{n}\left(\tilde{\phi}_j^2+2\tilde{w}_j\tilde{\phi}_j-\int\tilde{\phi}_j^2+2\tilde{w}_j\tilde{\phi}_j\right)\right)^2\leq c_5\sum_{i=m}^{n}\left(E[\tilde{\phi}_i]+\mathcal{O}(i^{-\tilde{k}})\right).$$

We choose $h_i = E[\tilde{\phi}_i]+\mathcal{O}(i^{-\tilde{k}})$, and so Proposition 8.1 implies that $\frac{1}{E_n}\sum_{j=1}^n\left(\tilde{\phi}_j^2\circ T^j+2(\tilde{w}_j\tilde{\phi}_j)\circ T^j\right)\to 1$ almost surely, provided $k\geq 2$. ∎

4 Applications to dynamical systems.

Theorem 3.1 applies to a variety of dynamical systems including Gibbs-Markov maps [1] and rational maps [17]. For Gibbs-Markov maps it has been shown [14, Theorem 1] that nested sequences of balls $(B_i(p))$ satisfy both the Strong Borel Cantelli property and Assumption (C), so that (I) applies. For rational maps [17, Theorem 10] shows that the transfer operator contracts exponentially in the $\mathscr{L}^\infty$ norm hence if the (SP) property is also proved then (II) holds. More generally (II) shows that proving the (SP) property for systems whose associated transfer operator has exponential decay suffices to prove the SBC property and the CLT for shrinking targets.

5 Decay in $BV(X)$ versus $\mathscr{L}^1$

It is known that summable decay of correlations in $BV(X)$ versus $\mathscr{L}^1$ implies the SP property by work of Kim [26, Proof of Theorem 2.1] (see also Gupta et al [14, Proposition 2.6]). Hence the statement in this setting is simpler.

Let the transfer operator P be defined by $\int \phi\psi \circ T\, d\mu = \int P\phi\psi\, d\mu$ for all $\phi,\ \psi \in \mathscr{L}^2(\mu)$, that is P is the adjoint of the Koopman operator $U\phi := \phi \circ T$.

We assume that the restriction of P to the space $BV(X)$ is exponentially contracting, i.e. $P : BV(X) \to BV(X)$ satisfies

$$\|P^n\phi\|_{BV} \le C\theta^n \|\phi\|_{BV} \tag{5.1}$$

for all $\phi \in BV(X)$ such that $\int \phi\, d\mu = 0$.

This implies that (T, X, μ) has exponential decay of correlations in BV versus $\mathscr{L}^1$, so that for some $0 < \theta < 1$,

$$\left|\int \phi\psi \circ T^n\, d\mu - \int \phi\, d\mu \int \psi\, d\mu\right| \le C\theta^n \|\phi\|_{BV} \|\psi\|_1 \tag{5.2}$$

for all $\phi \in BV(X)$, $\psi \in \mathscr{L}^1(\mu)$. In particular the measure μ is ergodic.

Proposition 5.1 *Assume the transfer operator P contracts exponentially as given by (5.1)*

Let $B_i := B(p, r_i)$ be nested balls of radius r_i about a point p such that $\sum_i \mu(B_i) = \infty$ and $\mu(B_i(p)) \leq \frac{C_2}{i^\gamma}$ for constants $C_2 > 0$ and $0 < \gamma \leq 1$. Let $a_n^2 = E(\sum_{j=1}^n (1_{B_i} \circ T^i - \mu(B_i)))^2$. Then:

(I)

$$\frac{1}{a_n} \sum_{j=1}^{n} (1_{B_i} \circ T^i - \mu(B_i)) \to N(0,1).$$

(II) If the nested sequence of balls $(B_i(p))$ about p satisfies Assumption (C) then

$$\frac{1}{a_n} \sum_{j=1}^{n} (1_{B_i} \circ T^i - \mu(B_i)) \to N(0,1)$$

in distribution.

Proof: The proof is the same as for Theorem 3.1 with the simplification that the SP property holds automatically as we have summable decay of correlations in $BV(X)$ versus $\mathscr{L}^1$ (see proof of [26, Theorem 2.1]). Furthermore Lemma 2.4 shows that the variance is unbounded and Lemma 2.9 gives a precise rate of growth in the case that Assumption (C) holds. ∎

Remark 5.2 For one-dimensional maps of the interval, Proposition 5.1 is basically a consequence of Conze and Raugi [6, Theorem 5.1]. Follow the proof of [6, Theorem 5.1] taking $T_k = T$ for all k, m to be the invariant measure μ and choosing $f_n = 1_{B_n}(p)$. The rates of growth are given by Lemma 2.4 which shows that the variance is unbounded. Lemma 2.9 gives a precise rate of growth in the case that Assumption (C) holds. In Proposition 6.2 we extend these results to piecewise expanding maps in higher dimensions.

6 Applications of Proposition 5.1.

Proposition 5.1 applies to certain classes of one-dimensional maps such as piecewise expanding maps of the interval $T : X \to X$ with $\frac{1}{T'}$ of bounded variation and possessing an absolutely continuous invariant measure with density bounded away from zero (those maps satisfying the assumptions of [26, Theorem 2.1], see also [14]). For these systems, Assumption (C) has been shown to hold for nested balls about μ a.e. $p \in X$ [21, 15]. In the next subsection we generalize these results to piecewise expanding maps in higher dimensions.

6.1 Piecewise expanding maps in higher dimensions

In this section we prove the Strong Borel Cantelli property and the CLT for shrinking balls in a class of expanding maps in higher dimensions. We also show that assumption C holds for μ-a.e. point.

The Banach spaces will be given by $\mathscr{L}^1$, defined with respect to the Lebesgue measure on $\mathbb{R}^n$, and a quasi-Hölder space with properties analogous to BV which we define below. A key property of the quasi-Hölder space is that characteristic functions of balls have bounded norm (as in the BV norm) which turns out to be a very useful property.

The maps are defined on compact sets $Z \in \mathbb{R}^N$. Denote by $\text{dist}(\cdot,\cdot)$ the usual metric in $\mathbb{R}^N$ and for $\varepsilon > 0$ let $B_\varepsilon(x) = \{y \in \mathbb{R}^N : \text{dist}(x,y) < \varepsilon\}$ be the ε-ball centred at x. Let $B_\varepsilon(A) = \{y \in \mathbb{R}^N : \text{dist}(y,A) \leq \varepsilon\}$ and write Z° for the interior of Z and $\overline{Z}$ its closure.

A map $T : Z \to Z$ is said to be a multidimensional piecewise expanding map, if there exists a family of finitely many disjoint open sets $\{Z_i\}$ such that $\text{Leb}(Z \setminus \bigcup_i Z_i) = 0$ and there exist open sets $\widetilde{Z}_i \supset \overline{Z_i}$ and $C^{1+\alpha}$ maps $T_i : \widetilde{Z}_i \to \mathbb{R}^N$ (for some $0 < \alpha \leq 1$) and some sufficiently small real number $\varepsilon_1 > 0$ such that for all i,

- (H1) $T_i(\widetilde{Z}_i) \supset B_{\varepsilon_1}(T(Z_i))$ and $T_i|_{Z_i} = T|_{Z_i}$;
- (H2) For $x, y \in T(Z_i)$ with $\text{dist}(x,y) \leq \varepsilon_1$,

$$|\det DT_i^{-1}(x) - \det DT_i^{-1}(y)| \leq c|\det DT_i^{-1}(x)|\text{dist}(x,y)^\alpha;$$

- (H3) There exists $s = s(T) < 1$ such that $\forall x, y \in T(\widetilde{Z}_i)$ with $\text{dist}(x,y) \leq \varepsilon_1$, we have

$$\text{dist}(T_i^{-1}x, T_i^{-1}y) \leq s\,\text{dist}(x,y).$$

- (H4) Let $G(\varepsilon) := \sup_x G(x, \varepsilon, \varepsilon_1)$ where

$$G(x, \varepsilon, \varepsilon_1) := \sum_i \frac{\text{Leb}(T_i^{-1} B_\varepsilon(\partial T Z_i) \cup B_{(1-s)\varepsilon_1}(x))}{\text{Leb}(B_{(1-s)\varepsilon_1}(x))} \tag{6.1}$$

and assume that

$$\sup_{\delta \leq \varepsilon_1} \left(s^\alpha + 2 \sup_{\varepsilon \leq \delta} \frac{G(\varepsilon)}{\varepsilon^\alpha} \delta^\alpha \right) < 1^1 \tag{6.2}$$

We now introduce the Banach space of quasi-Hölder functions in which the spectrum of the Perron-Frobenius operator P is investigated. Given a Borel set $\Gamma \subset Z$, we define the oscillation of $\varphi \in \mathscr{L}^1(\text{Leb})$ over Γ as

$$\text{osc}(\varphi, \Gamma) := \operatorname*{ess\,sup}_\Gamma \varphi - \operatorname*{ess\,inf}_\Gamma \varphi.$$

The function $x \mapsto \text{osc}(\varphi, B_\varepsilon(x))$ is measurable (see [23, Proposition 3.1]) For $0 < \alpha \leq 1$ and $\varepsilon_0 > 0$, we define the α-seminorm of φ as

$$|\varphi|_\alpha = \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{-\alpha} \int_{\mathbb{R}^N} \text{osc}(\varphi, B_\varepsilon(x))\, \text{dLeb}(x).$$

Let us consider the space of functions with bounded α-seminorm

$$V_\alpha = \{\varphi \in \mathscr{L}^1(\text{Leb}) : |\varphi|_\alpha < \infty\},$$

and endow V_α with the norm

$$\| \cdot \|_\alpha = \| \cdot \|_1 + | \cdot |_\alpha$$

[1] This condition could be greatly simplified as follows. Suppose the boundaries of Z_i are C^1 codimension one embedded compact submanifold, then define the quantity:

$$\eta_0(T) := s^\alpha + \frac{4s}{1-s} Y(T) \frac{\xi_{N-1}}{\xi_N}$$

where

$$Y(T) = \sup_x \sum_i \# \{\text{smooth pieces intersecting } \partial Z_i \text{ containing } x\},$$

is the maximal number of smooth components of the boundaries that can meet in one point and $\xi_N = \frac{\pi^{N/2}}{(N/2)!}$, the N-volume of the N-dimensional unit ball of $\mathbb{R}^N$. We require that $\eta_0(T) < 1$, and this may replace the condition (6.2) above.

which makes it into a Banach space. We note that V_α is independent of the choice of ε_0 and that V_α is continuously injected in $\mathscr{L}^\infty(\text{Leb})$. According to [34, Theorem 5.1], there exists an absolutely continuous invariant probability measure (a.c.i.p.) μ, with density bounded above, and bounded below from zero, which has exponential decay of correlations against $\mathscr{L}^1$ observables on the finitely many mixing components of V_α: in view of the next Theorem 6.4 we will from now restrict ourselves to one of those components, by taking a mixing iterate of T. More precisely, if the map T is as defined above and if μ is the mixing a.c.i.p., then there exist constants $C < \infty$ and $\vartheta < 1$ such that

$$\Big|\int_Z \psi \circ T^n \, h \, \mathrm{d}\mu - \int \psi \mathrm{d}\mu \int h \mathrm{d}\mu\Big| \leq C \|\psi\|_{\mathscr{L}^1} \|h\|_\alpha \vartheta^n \tag{6.3}$$

for all $\psi \in \mathscr{L}^1$ and for all $h \in V_\alpha$. Moreover $\|P^n \phi\|_\alpha \leq C\|\phi\|_\alpha$ for all $\phi \in V_\alpha$ and thus equation 5.1 holds.

We now show that characteristic functions of balls are bounded in the $\|\cdot\|_\alpha$ norm.

Lemma 6.1 *Let $B_i(p)$ be a nested sequence of balls about a point $p \in X$, then there exists a constant $D(\alpha)$ such that*

$$\|1_{B_i}\|_\alpha \leq D(\alpha)$$

for all i.

Proof: Take any set A with a rectifiable boundary. If p is not in a 2ϵ neighborhood of the boundary of A, then the oscillation is zero, otherwise it is 1. Therefore we have $\int osc(1_A, B_\epsilon(p)) \, \mathrm{dLeb}(p) \leq c_1 \epsilon$. Then we must divide by ϵ^α. As $\alpha \leq 1$ we have the ratio bounded by $c_1 * \epsilon^{1-\alpha}$. ∎

The boundedness of the characteristic functions in the $\|\cdot\|_\alpha$-norm allows us to proceed as in Proposition 5.1 (see also [6]) and to obtain the following result.

Proposition 6.2 *Assume a piecewise expanding map T on a compact set $Z \subset \mathbb{R}^n$ satisfies conditions (H1)–(H4) and is mixing with respect to its absolutely continuous invariant measure μ. Let $B_i := B(p, r_i)$ be balls of radius r_i about a point p such* *that* $\sum_i \mu(B_i) = \infty$ *and* $\mu(B_i) \le \frac{C_2}{i^\gamma}$ *for some constants* C_2 *and* $\gamma \in (0,1]$. *Then the variance* $a_n^2 := E[(\sum_{j=1}^n (1_{B_i} \circ T^i - \frac{1}{i}))^2]$ *satisfies*

$$\limsup_{n\to\infty} \frac{a_n}{\sqrt{\log n}} \ge 1$$

and

$$\frac{1}{\sqrt{a_n}} \sum_{j=1}^n (1_{B_i} \circ T^i - \frac{1}{i}) \to N(0,1)$$

in distribution.

Proof: The SBC property (I) is immediate from the decay of correlations, Equation 6.3 and the bound $\|1_{B_i}\|_\alpha \le D(\alpha)$ by the proof of Proposition 6.1. The growth estimate follows from Lemma 2.4. ∎

We now make an additional assumption. Suppose that we have M domains of local injectivity for the map T; if we take the join $\mathcal{Z}^j := \bigvee_{i=0}^{j-1} T^{-i}\mathcal{Z}$, where $\mathcal{Z}$ denotes the partition, mod-0, into the closed sets $\overline{Z_i}, i = 1, \cdots, M$, then on each element $Z_l^{(j)}, l = 1, \cdots, |\mathcal{Z}^j|$, each of which is the closure of its interior, the map T^j is injective and of class $C^{1+\alpha}$ on an open neighborhood of $Z_l^{(j)}$: we call $\tilde{Z}_l^{(j)}$ such an extension. In order to prove condition (C) we require a further assumption which is also called the *finite range structure*. We assume:

- (H5) Let $\mathcal{U}^{(j)} := \{T^j Z_l^{(j)}, \forall l = 1, \cdots, |\mathcal{Z}^j|\}$, and put $\mathcal{U} = \cup_{j=1}^\infty \mathcal{U}^{(j)}$. Then $\mathcal{U}$ consists of only finitely many subsets of Z with positive Lebesgue measure, hence $U_m = \inf_{U\in\mathcal{U}} m(U)$ is bounded below.

Lemma 6.3 *Under the assumptions (H1)–(H5) Assumption (C) is satisfied.*

Proof. Denote

$$\mathscr{E}_k(\varepsilon) := \{x; \mathrm{dist}(T^k x, x) \le \varepsilon\}.$$

By Lemma 8.2 (see Appendix) it is enough to prove that there exists $C > 0$, $\delta > 0$ such that for all k and ε,

$$\mu(\mathscr{E}_k(\varepsilon))) < C\varepsilon^\delta.$$

We now fix j and consider the cylinder, say, $Z_l^{(j)}$. Let us suppose that $\{z_k\}_{k\geq 1}$ is a sequence of points in $Z_l^{(j)}$ converging to $x \in Z_l^{(j)}$, namely $\text{dist}(z_k, x) \to 0$ when $k \to \infty$, and that $\text{dist}(T^j(z_k), x) \to 0$ for $k \to \infty$. With abuse of definition we say that such a point x is *fixed*. If there are points in the sequence $\{z_k\}_{k\geq 1}$ which are on the boundary of $Z_l^{(j)}$, we think of T^j as its $C^{1+\alpha}$ extension on $\tilde{Z}_l^{(j)}$. We want to show that in $\tilde{Z}_l^{(j)}$ there is only one fixed point x. By contradiction, suppose y is another fixed point and $\{w_k\}_k$ a sequence converging to y and whose T^j images converge to y as well. Suppose that $\tilde{Z}_l^{(j)}$ is a convex set in such a way the segment $[x, y]$ is contained in $\tilde{Z}_l^{(j)}$ [2]. We now fix η small enough and take k big enough and such that $\text{dist}(x, z_k)$, $\text{dist}(x, T^j(z_k))$, $\text{dist}(y, w_k)$, $\text{dist}(y, T^j(w_k))$, are all smaller than η. We also put $D_{m,j} := \inf\{||DT^j(x)||\} > 1$, where the inf is taken over the points x where the derivative is defined. The norm is the operator norm, which is strictly larger than 1 since the map is uniformly expanding. Then we have

$$\text{dist}(x, y) \geq \text{dist}(T^j(z_k), T^j(w_k)) - \text{dist}(x, T^j(z_k)) - \text{dist}(y, T^j(w_k))$$

and by applying Taylor's formula

$$\text{dist}(x, y) \geq D_{m,j}\text{dist}(z_k, w_k) - 2\eta \geq D_{m,j}[\text{dist}(x, y) - 2\eta] - 2\eta$$

which gives a contradiction, since $D_{m,j} > 1$, by sending η to 0. Hence x is the only fixed point.

Let us now take a measurable set $V \subset \tilde{Z}_l^{(j)}$ containing the fixed point $x \in \tilde{Z}_l^{(j)}$. We require that the diameter of the image $T^j(V)$ be at most ε; such an image will therefore be contained in the ball of center $T^j(x)$ and of radius ε. The Lebesgue measure of this ball will be equal to $\xi_N \varepsilon^N$, where the factor ξ_N was defined in the preceding footnote. Then we have

$$\text{Leb}(B_\varepsilon(x)) = \xi_N \varepsilon^N \geq \text{Leb}(T^j(V)) \geq |\det(DT^j(\kappa))|\text{Leb}(V)$$

[2] If not we could join x and y with a chain of segments contained each in $\tilde{Z}_l^{(j)}$: the argument will work again since the sum of the lengths of those segments is larger than the distance between x and y and this is what we need in bounding from below.

for a suitable point $\kappa \in \tilde{Z}_l^j$, where in the last inequality we used a local change of variable and the continuity of DT^j. By distortion, we could replace this point by another one, say ι such that $\text{Leb}(T^j(Z_l^{(j)}) = |\det(DT^j(\iota))|\text{Leb}(Z_l^{(j)})$; we call B the distortion constant satisfying $\frac{|\det(DT^j(\iota))|}{|\det(DT^j(\kappa))|} \leq B$. We therefore get

$$\text{Leb}(V) \leq \frac{\xi_N \varepsilon^N B}{|\det(DT^j(\iota))|} \leq \frac{\xi_N \varepsilon^N B \ \text{Leb}(Z_l^{(j)})}{U_m}$$

Since the density of the absolutely continuous invariant measure μ is bounded from above (remember it is in $\mathscr{L}^\infty(\text{Leb})$), by, say, h_M, and since each $Z_l^{(j)}$ will contribute with at most one fixed point, by taking the sum over the l we will equivalently get an upper bound on the total measure of the balls including the $T^j(V)$; hence we finally get

$$\mu\{x; \text{dist}(T^j x, x)\} \leq \frac{\xi_N h_M \ \varepsilon^N B}{U_m}.$$

and this bound is independent of j. ∎

As a consequence of Lemma 2.8 we have,

Theorem 6.4 *Assume a piecewise expanding map T on a compact set $Z \subset \mathbb{R}^n$ satisfies conditions (H1)–(H5) and is mixing with respect to its absolutely continuous invariant measure μ. For μ a.e. p if $B_i(p)$ are nested balls about p such that $\sum_i \mu(B_i) = \infty$ and $\mu(B_i) \leq \frac{C_2}{i^\gamma}$ for some constants $C_2 > 0$ and $\gamma \in (0, 1]$. Then*

$$a_n^2 = E[(\sum_{j=1}^n (1_{B_i} \circ T^i - \frac{1}{i}))^2] = \log n + \mathcal{O}(1)$$

and

$$\frac{1}{\sqrt{\log n}} \sum_{j=1}^n (1_{B_i} \circ T^i - \frac{1}{i}) \to N(0, 1)$$

in distribution.

7 Discussion.

There are several natural questions remaining unanswered. In particular can the CLT for shrinking targets be proved for Anosov systems or non-uniformly hyperbolic diffeomorphisms? Chernov and Kleinbock have proved the SBC property for balls in Anosov systems [4] but the SBC property is unknown for non-uniformly hyperbolic diffeomorphisms. More generally can a limit theory be developed for the statistics of non-stationary stochastic processes arising as observations (which change in time) on deterministic dynamical systems which may also may evolve in time, such as sequential dynamical systems?

8 Appendices

8.1 Gal-Koksma Theorem.

We recall the following result of Gal and Koksma as formulated by W. Schmidt [35, 36] and stated by Sprindzuk [38]:

Proposition 8.1 *Let* $(\Omega, \mathcal{B}, \mu)$ *be a probability space and let* $f_k(\omega)$, $(k = 1, 2, \ldots)$ *be a sequence of non-negative* μ *measurable functions and* g_k, h_k *be sequences of real numbers such that* $0 \le g_k \le h_k \le 1$, $(k = 1, 2, \ldots,)$. *Suppose there exists* $C > 0$ *such that*

$$\int \left(\sum_{m<k\le n} (f_k(\omega) - g_k) \right)^2 d\mu \le C \sum_{m<k\le n} h_k \qquad (*)$$

for arbitrary integers $m < n$. *Then for any* $\epsilon > 0$

$$\sum_{1\le k\le n} f_k(\omega) = \sum_{1\le k\le n} g_k + O(\Theta^{1/2}(n) \log^{3/2+\epsilon} \Theta(n))$$

for μ *a.e.* $\omega \in \Omega$, *where* $\Theta(n) = \sum_{1\le k\le n} h_k$.

8.2 Assumption (C) for expanding systems

In this appendix we show that if the invariant measure μ has a density $\rho(x)$ with respect to Lebesgue measure m then Assumption (C) is valid. Recall we define

$$\mathscr{E}_k(\epsilon) := \{x : d(T^k x, x) \le \epsilon\}$$

Lemma 8.2 *Let $B_i(x)$ denote a decreasing sequence of balls with center x and suppose $\mu(B_i(x)) \leq \frac{C_2}{i^\gamma}$ for some constants C_1, C_2 and $0 < \gamma \leq 1$. Suppose μ has a density ρ with respect to Lebesgue measure m with support X and there exists $C_7 > 0$, $\delta > 0$ such that for all k, ϵ,*

$$\mu(\mathscr{E}_k(\epsilon)) < C_7 \epsilon^\delta$$

Then for μ a.e. $p \in X$ there exists $\eta(p) \in (0,1)$ and $\kappa > 1$ such that for all i sufficiently large

$$\mu(B_i(p) \cap T^{-r} B_i(p)) \leq \mu(B_i(p))^{1+\eta}$$

for all $r = 1, \ldots, \log^\kappa i$.

Proof. Let $\rho(x) = \frac{d\mu}{dm}(x)$ be the density of μ with respect to Lebesgue measure m.

Let $\sigma \geq 1$ and $\gamma > \sigma$. We choose ϵ_k so that for all x a ball of radius ϵ_k about x, denoted $B(x, \epsilon_k)$, satisfies $c_1/k^\sigma \leq m(B(x, \epsilon_k)) \leq c_2/k^\sigma$, so $\epsilon_k \simeq k^{-\sigma/D}$ where D is the dimension of X and $c_1, c_2 > 0$.

Let $\kappa > 1$ and define $A_k := \{x : d(T^j x, x) \leq \epsilon_k \text{ for some } 1 \leq j \leq \log^\kappa k\}$. Evidently $A_k \subset \bigcup_{j=1}^{\log^\kappa k} \mathscr{E}_j$. By the estimate on $\mathscr{E}_k(\epsilon)$ for all large k, $\mu(A_k) \leq c_3 \epsilon_k^\tau$ where $\tau < \delta$. Let

$$F_k := \{x : \mu(B(x, \epsilon_k) \cap A_k) \geq 1/k^\gamma\}$$

and define the Hardy-Littlewood maximal function M_k for $\phi(x) = 1_{A_k}(x)\rho(x)$ by

$$M_k(x) := \sup_{a>0} \frac{1}{m(B_a(x))} \int_{B_a(x)} 1_{A_k}(y)\rho(y)\, dm(y).$$

If $x \in F_k$ then $M_k > c_2^{-1} k^{\sigma-\gamma}$.

A theorem of Hardy and Littlewood ([10] Theorem 3.17) states that

$$m(|M_k| > C) \leq c_4 \frac{\|1_{A_k}\rho\|_1}{C}$$

for some constant c_4, where $\|\cdot\|_1$ is the $\mathscr{L}^1$ norm with respect to m. Hence

$$m(F_k) \leq m(M_k > c_2^{-1} k^{\sigma-\gamma}) \leq c_4 \mu(A_k) c_2 k^{\gamma-\sigma} \leq k^{\gamma-\sigma(1+\tilde{\tau})}$$

where $0 < \tilde{\tau} < \tau/D$. We need to alter τ/D to $\tilde{\tau}$ to take into account the fact that a ball of radius ϵ has measure roughly ϵ^D.

Choosing σ large enough that $\sigma\tilde{\tau} > 1$ and then taking $\sigma < \gamma < \sigma - 1 + \sigma\tilde{\tau}$ the series $\sum_k m(F_k)$ converges.

So for m a.e. x_0 there exists an $N(x_0)$ such that $x_0 \notin F_k$ for all $k > N(x_0)$. Hence for $k > N(x_0)$, $\mu(B(x,\epsilon_k) \cap A_k) \leq 1/k^\gamma$, thus $\mu(B(x,\epsilon_k) \cap A_k) \leq m(B(x,\epsilon_k))^{1+\eta}$ for some $\eta > 0$ (recall $m(B(x,\epsilon_k)) \simeq \frac{1}{k^\sigma}$ and $\gamma > \sigma$).

Furthermore by the Lebesgue differentiation theorem for m a.e. x

$$\lim_{\epsilon \to 0} \frac{1}{m(B_\epsilon(x))} \int_{B_\epsilon(x)} \rho(y) dm = \rho(x)$$

and for μ a.e. x , $\rho(x) > 0$ as X is the support of μ. Hence for m a.e. x_0 there exists an $\tilde{N}(x_0)$ and $\tilde{\eta} > 0$ such that for all $k > \tilde{N}(x_0)$ we have $\mu(B(x,\epsilon_k) \cap A_k) \leq \mu(B(x,\epsilon_k))^{1+\tilde{\eta}}$.

As κ was arbitrary by interpolating between the sequence ϵ_k we have that for μ a.e. $x \in X$ there exists $\eta' > 0$, $\kappa' > 1$ such that

$$\mu(B_i(x) \cap T^{-r} B_i(x)) \leq \mu(B_i(x))^{1+\eta'}$$

for $1 \leq r \leq \log^{\kappa'} i$. This is Assumption (C). ∎

Acknowledgements

We would like to thank the anonymous referees whose careful reading and perceptive comments helped us improve the final version of this work. MN and LZ were partially supported by the National Science Foundation under Grant Number DMS-1101315. Part of this work was done while MN was visiting the *Centre de Physique Théorique* in Marseille with French CNRS support. SV was supported by the CNRS-PEPS *Mathematical Methods of Climate Theory*, by the ANR-Project *Perturbations* and by the PICS (Projet International de Coopération Scientifique), *Propriétés statistiques des systèmes dynamiques detérministes et aléatoires*, with the University of Houston, n. PICS05968. Part of this work was done while he was visiting the *Centro de Modelamiento Matemático, U MI2807*, in Santiago de Chile with a CNRS support (délégation).

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