Dynamical Borel-Cantelli lemmas and rates of growth of Birkhoff sums of non-integrable observables on chaotic dynamical systems.

Meagan Carney [*] M. Nicol [†]

May 5, 2017

Abstract

We consider implications of dynamical Borel-Cantelli lemmas for rates of growth of Birkhoff sums of non-integrable observables $\varphi(x) = d(x,q)^{-k}$, $k > 0$, on ergodic dynamical systems (T, X, μ) where $\mu(X) = 1$. Some general results are given as well as some more concrete examples involving non-uniformly expanding maps, intermittent type maps as well as uniformly hyperbolic systems.

1 Birkhoff sums of non-integrable functions.

Let X_i be a sequence of random variables on a probability space (X, μ) (in other words a stochastic process) and let $S_n = \sum_{i=1}^{n} X_i$ be the associated sequence of Birkhoff sums.

W. Feller [9] showed that if $\{X_i\}$ are iid and $E|X_1| = \infty$ then for any sequence $b(n) > 0$, if $\lim_{n\to\infty} \frac{b(n)}{n} = \infty$ then either $\limsup \frac{S_n}{b(n)} = \infty$ a.e. or $\liminf \frac{S_n}{b(n)} = 0$ a.e. Chow and Robbins [7] then showed that the conditions on $b(n)$ can be relaxed and that in fact for any sequence of constants $b(n)$ either $\limsup \frac{S_n}{b(n)} = \infty$ a.e. or $\liminf \frac{S_n}{b(n)} = 0$ a.e.

[*]Department of Mathematics, University of Houston, Houston, USA. e-mail: <meagan@math.uh.edu>
Meagan Carney thanks the NSF for partial support on NSF-DMS Grant 1600780.

[†]Department of Mathematics, University of Houston, Houston Texas, USA. e-mail: <nicol@math.uh.edu>. Matthew Nicol thanks the NSF for partial support on NSF-DMS Grant 1600780 and the hospitality and support of the Max Planck Institute for the Physics of Complex Systems, Dresden, where this work was completed. Both authors thank Alan Haynes for enlightening conversations and the two anonymous referees for their helpful comments.

Suppose now that (T, X, μ) is an ergodic probability measure preserving transformation and $\varphi : X \to \mathbb{R}$ is a non-integrable measurable function. Then $X_i := \varphi \circ T^i$ is a stationary stochastic process with Birkhoff sum $S_n = \sum_{i=1}^n \varphi \circ T^i$. In this dynamical setting Aaronson [1] showed that for any sequence $b(n) > 0$, if $\lim_{n\to\infty} \frac{b(n)}{n} = \infty$ then either $\limsup \frac{S_n}{b(n)} = \infty$ a.e. or $\liminf \frac{S_n}{b(n)} = 0$ a.e. Thus for ergodic dynamical systems there is no strong law of large numbers for non-integrable observables.

A natural question is the rate of growth of Birkhoff sums. A useful result, due again to Aaronson [1, Proposition 2.3.1] states:

Proposition 1.1 *Suppose that $\varphi : X \to \mathbb{R}$ is a non-integrable measurable function. If $a(x)$ is increasing, $\lim_{x\to\infty} \frac{a(x)}{x} = 0$ and*

$$\int a(|\varphi(x)|) d\mu < \infty$$

then for μ a.e. x

$$\lim_{n\to\infty} \frac{a(|S_n|)}{n} = 0$$

Despite the generality of its assumptions, the above gives close to optimal bounds on $\limsup S_n$ in many dynamical settings, as demonstrated later in this paper. Throughout this paper if $a(n)$ and $b(n)$ are two sequences $a(n) \sim b(n)$ will mean that there exists an N and constants C_1, C_2 such that $0 < C_1 \le \frac{a(n)}{b(n)} \le C_2$ for all $n \ge N$.

In [15] dynamical Borel-Cantelli lemmas were used to give information on the almost sure behavior of the maxima $M_n := \max\{\varphi(x), \varphi(Tx), \varphi(T^2x), \ldots, \varphi(T^n x)\}$ of a time-series for certain classes of observables on a variety of chaotic dynamical systems (T, X, μ). Motivated by applications in extreme value theory the observables considered in [15] were of form $\varphi(x) = -\log d(x, q)$ and $\varphi(x) = d(x, q)^{-k}$ for a point $q \in X$, where $d(.,.)$ was a Riemannian metric on the space X, a Riemannian manifold (in applications an open subset of Euclidean space). For the integrable observable $\varphi(x) = -\log d(x, q)$ under relatively mixing conditions on the dynamical system (please see [15, Theorem 2.2] for details) a sequence of scaling constants $a(n)$ exists such that $\lim_{n\to\infty} \frac{M_n}{a(n)} = C > 0$ almost surely for some constant C.

But the following result shows that for many dynamical systems there is no almost sure limit for $\frac{M_n}{a(n)}$, in the case $\varphi(x) = d(x, q)^{-k}$, $k > 0$, even if k is such that φ is integrable (so that a strong law of large numbers does hold for the Birkhoff sum). We state a simpler, less general, version of [15, Theorem 2.7] adapted for our purposes,

Proposition 1.2 *Suppose that (T, X, μ) is a probability measure preserving system with ergodic measure μ which is absolutely continuous with respect to Lebesgue measure m. Suppose for a point $q \in X$ there exists $\delta > 0$, $C > 0$ and $r_0 > 0$ such that for all $\varepsilon < r < r_0$:*

$$|\mu(B(q, r + \varepsilon)) - \mu(B(q, r))| \leq C\varepsilon^{\delta}. \tag{1.1}$$

and $0 < \frac{d\mu}{dm}(q) < \infty$ where $B(q, r)$ denotes the ball of radius r centered at q. Moreover suppose that we have exponential decay of correlations in bounded variation norm (BV) versus L^1 in the sense that there exists $C > 0$ and $0 < \theta < 1$ such that for all φ_1 of bounded variation and all $\varphi_2 \in L^1(m)$ we have:

$$\left| \int \varphi_1 \cdot \varphi_2 \circ T^j d\mu - \int \varphi_1 d\mu \int \varphi_2 d\mu \right| \leq C\theta^j \|\varphi_1\|_{BV} \|\varphi_2\|_{L^1(m)},$$

Then if $\varphi(x) = d(x, q)^{-k}$ for some $k > 0$ for any monotone sequence $u(n) \to \infty$:

$$\mu\left(\limsup_{n\to\infty} \frac{M_n(x)}{u(n)} = 0\right) = 1, \text{ or } \mu\left(\liminf_{n\to\infty} \frac{M_n(x)}{u(n)} = \infty\right) = 1. \tag{1.2}$$

The relation between Birkhoff sums and extreme values, such as the maxima, is investigated in the topic of trimmed Birkhoff sums [3, 16, 28]. In this approach the time series $\{\varphi(x), \varphi(Tx), \varphi(T^2x), \ldots, \varphi(T^n x)\}$ is rearranged into increasing order $\{\varphi(T^{i_0}x) \leq \varphi(T^{i_1}x) \leq \varphi(T^{i_2}x) \leq \ldots \varphi(T^{i_n}x)\}$ so that $\varphi(T^{i_n}x) = M_n(x)$. We will this denote this rearrangement by $\{M_0^n(x), M_1^n(x), \ldots, M_n^n(x)\}$. Note that $M_n(x) = M_n^n(x)$ in this notation. Almost sure limit theorems for trimmed sums involve two sequences of constants $a(n)$,$b(n)$ so that the scaled truncated sum $\frac{1}{a(n)} \sum_{j=0}^{n-b(n)} M_j^n$ satisfies a strong law of large numbers. We refer especially to [3, 28], where very precise information on the limiting behavior and choice of constants $a(n)$,$b(n)$ is given for certain dynamical systems. Such results make clear the relations between large extremal values of the time series and the behavior of the Birkhoff sum. There still remains the question of the rate of growth of $\sum_{j=n-b(n)+1}^{n} M_j^n$. However good estimates on the lower bound of the rate of growth of S_n are given by the constants $a(n)$ in the trimmed sum limit. In fact [28, Theorem 1.8] provides a better bound for $\liminf S_n$ in the context of piecewise uniformly expanding interval maps than our techniques. We remark on this at more length later.

In this paper we will consider the observable $\varphi(x) = d(x, q)^{-k}$ over chaotic dynamical systems (T, X, μ) for values of k which ensure that $\int \varphi d\mu = \infty$. Our results are limited to probability spaces, in that $\mu(X) = 1$. Most of our results generalize in an obvious way to a wider class of functions, for example those for which $\mu(\varphi > t) = \frac{L(t)}{t^\gamma}$ where $0 < \gamma < 1$ and $L(t)$ is a slowly varying function, as long as the sets $(\varphi > t)$ for large t correspond to sets for which dynamical Borel-Cantelli lemmas hold. Similarly our results generalize to observables φ with a finite set of singularities $\{q_1, \ldots, q_m\}$ such that for all i there exist constants C_1, C_2 $r > 0$ such that $0 < C_1 < \frac{\varphi(y)}{d(y,q_i)^{-k}} < C_2$ for all $y \in B(q_i, r)$ and with integrable negative part i.e. if $\varphi^- := \max\{0, -\varphi\}$ then $\int \varphi^- d\mu < \infty$. But for simplicity of exposition we will stick to the $\varphi(x) = d(x,q)^{-k}$. The case where $\varphi^-(x)$ is not integrable is very interesting but the techniques of this paper are not immediately applicable to this case. We refer to [2, 19] for interesting recent results on trimmed symmetric Birkhoff sums in the setting of infinite ergodic theory (when the underlying probability space has infinite measure).

2 Dynamical Borel Cantelli lemmas and infinite Birkhoff sums.

We assume that (T, X, μ) is an ergodic dynamical system and X is a probability measure space and a Riemannian manifold with a Riemannian metric d. Let m denote Lebesgue measure on X and assume $\mu \ll m$. Let $B(q,r) := \{x : d(q,x) < r\}$ denote the ball of radius r about a point q with respect to the given metric d. Suppose that B_j is a sequence of nested balls in X based about a point q. Define

$$E_n = \sum_{j=1}^{n} \mu(B_j)$$

For the purposes of this paper (see [6], who introduced the term) we say that the Strong Borel Cantelli (SBC) property holds for (B_j) if for μ a.e. $x \in X$

$$\sum_{j=1}^{n} 1_{B_j} \circ T^j(x) = E_n + o(E_n)$$

In most of the examples we consider we have a better estimate of the error term and, for any $\delta > 0$,

$$\sum_{j=1}^{n} 1_{B_j} \circ T^j(x) = E_n + O(E_n^{1/2+\delta}) \tag{*}$$

If $(*)$ holds we say that the sequence (B_j) satisfies the QSBC property, for quantitative Strong Borel Cantelli property. If $T^j(x) \in B_j$ infinitely often for μ a.e. x we say that the sequence (B_j) has the Borel-Cantelli property.

Examples of systems for which the QSBC property has been proved for balls nested at points q in phase space include Axiom A diffeomorphisms [6], uniformly partially hyperbolic systems preserving a volume measure with exponential decay of correlations [8], uniformly expanding C^2 maps of the interval [26], and Gibbs-Markov type maps of the interval [18] (we define precisely Gibbs-Markov maps in the appendix). For intermittent type maps with an absolutely continuous invariant probability measure the work of Kim [18] and Gouëzel [12] gives a fairly complete picture: the Borel-Cantelli property holds for nested balls except those based at the indifferent fixed point. Other results on non-uniformly expanding systems include one-dimensional maps modeled by Young towers with exponential decay of correlations [13], the general framework of [14] and other hyperbolic settings [11, 20, 24, 17].

2.1 Non-integrable observations.

Let $\varphi(x) = d(x,q)^{-k}$ for some distinguished point q, where $\dim(X) = D$ and $k \geq D$. Let $S_n = \sum_{j=1}^{n} \varphi \circ T^j(x)$. Throughout this paper, in light of the examples we discuss, we will only consider phase spaces X which are Riemanian manifolds.

Theorem 2.1 *Suppose that (T, X, μ) is an ergodic dynamical system where X is a Riemannian manifold with Riemannian metric d. Suppose $\dim(X) = D$. Let $\varphi(x) = d(x,q)^{-k}$ for some distinguished point q. Suppose there exist constants C_1, C_2 such that $0 < C_1 < \frac{d\mu}{dm}(q) < C_2$ and that the SBC property holds for nested balls about q.*

If $k > D$ then for μ a.e. x and any $\varepsilon > 0$

$$(a) \;\; \limsup_{n\to\infty} \frac{S_n}{n^{k/D}[\log(n)]^{k/D+\varepsilon}} = 0$$

and for any $\varepsilon > 0$

$$(b) \;\; \liminf_{n\to\infty} \frac{S_n}{n^{k/D-\varepsilon}} = \infty$$

while

$$(c) \;\; S_n \geq n^{k/D} \log^{k/D} n \text{ infinitely often}$$

If moreover the QSBC property holds for nested balls about q then for any $\varepsilon > 0$

$$(d) \;\; \liminf_{n\to\infty} \frac{S_n}{n^{k/D}(e^{-(\log n)^{\frac{1}{2}+\varepsilon}})^{k/D}} = \infty$$

If $k = D$ the lower bounds in (b) and (d) may be replaced by $\liminf \frac{S_n}{n} = \infty$ *while (a) and (c) hold.*

Proof.

We assume first $k > D$. It is known from Aaronson [1, Proposition 2.3.1] that if $a(x)$ is increasing, $\lim_{x\to\infty} \frac{a(x)}{x} = 0$ and

$$\int a(\varphi(x))d\mu < \infty$$

then for μ a.e. x

$$\lim_{n\to\infty} \frac{a(S_n)}{n} = 0$$

Our assumptions imply that $\mu(B(q,r)) \sim r^D$. In fact using spherical coordinates our assumption on the density implies that for any integral $f : X \to \mathbb{R}$ $\int f d\mu = \int f h(x)dx = \int f(\theta_1, \ldots, \theta_{D-1}, r)K(\theta_1, ..., \theta_{D-1})r^{D-1}h(r)dr\theta_1 \ldots d\theta_{D-1}$ where $0 < c_1 < K(\theta_1, ..., \theta_{D-1}) < c_2$ for some constants c_1,c_2.

By the Borel Cantelli lemma $\mu(T^n x \in B(q, \frac{1}{n^{1/D+\delta}})\ i.\ o.) = 0$ for any $\delta > 0$. Hence given $\delta > 0$ for μ a.e. $x \in X$ there exists a time $N(x)$ such that $T^i x \notin B(q, \frac{1}{n^{1/D+\delta}})$ for all $i > N(x)$. This implies that $\varphi \circ T^j \le n^{k(1/D+\delta)}$ for all $j \ge N(x)$. Thus $S_n \le C(x)n^{1+k(1/D+\delta)}$ for large n where $C(x)$ is a constant. Hence $\log(S_n) \le c(x)\log(n)$ for some constant $c(x) > 0$. Choosing $a(x) = \frac{x^{D/k}}{\log(x)^{1+\eta}}$ for $\eta > 0$ then

$$a(S_n) = \frac{(S_n)^{D/k}}{\log(S_n)^{1+\eta}} \ge \frac{(S_n)^{D/k}}{[c(x)\log(n)]^{1+\eta}}$$

Hence for any $\varepsilon > 0$

$$\limsup \frac{S_n}{n^{k/D}[\log(n)]^{k/D+\varepsilon}} = 0$$

Assume now that the SBC property holds for nested balls about q. First note that if $r_n = (n)^{-1/D}$ then $T^n x \in B(q, r_n)$ i.o. Let $B_j := B(q, \frac{1}{j^{1/D}})$. From the SBC property $\sum_{j=1}^n 1_{B_j} \circ T^j(x) \sim \log(n)$.

If we define $n_l := \max\{0 < j \le n\}$ such that $T^j x \in B(q, r_j)$ (the notation "l" in n_l suggests the "last time") then for μ a.e. $x \in X$, for any $M > 0$, $\lim_{n\to\infty} \frac{n_l}{n^{1-\delta}} > M$ for any $\delta > 0$. To see this, for a generic $x \in X$, $\lim_{n\to\infty} \frac{S_n}{\log n} = 1$. By definition of $n_l(x)$, $S_{n_l} = S_n$ and hence $\lim_{n\to\infty} \frac{S_{n_l}}{\log n} = 1$. As $\lim_{n\to\infty} \frac{S_{n_l}}{\log n_l} = 1$ we see $\lim_{n\to\infty} \frac{\log n_l}{\log n} = 1$, which implies the result.

Since $S_n > M_{n_l}$, $\liminf \frac{S_n}{n^{k/D-\varepsilon}} = \infty$ for any $\varepsilon > 0$.

Suppose now that we have a quantitative error estimate in the form of the QSBC property,

$$\sum_{j=1}^{n} 1_{B_j} \circ T^j(x) = E_n + O(E_n^{1/2+\delta})$$

Then

$$S_n = E_n + O(E_n^{1/2+\delta})$$

$$S_{n_l} = E_{n_l} + O(E_{n_l}^{1/2+\delta})$$

By definition of n_l, $S_{n_l} = S_n$ and hence

$$E_n - E_{n_l} = O(E_n^{1/2+\delta})$$

We obtain

$$\log n - \log n_l = O(E_n^{1/2+\delta})$$

which implies that

$$n_l \geq n e^{-(\log n)^{\frac{1}{2}+\delta}}$$

for any $\delta > 0$.

Hence $\liminf \frac{S_n}{n^{k/D}(e^{-(\log n)^{\frac{1}{2}+\varepsilon}})^{k/D}} = \infty$ for any $\varepsilon > 0$.

The proofs of (a) and (c) in the case $k = D$ are unchanged, and estimates (b) and (d) are immediate consequences of the ergodic theorem. ∎

Remark 2.2 The assumptions of Theorem 2.1 are satisfied by Anosov diffeomorphisms [6], uniformly expanding C^2 maps of the interval [26] and Gibbs-Markov type maps of the interval [18]. Kim also shows that for all $q \in (0, 1]$ in a class of intermittent maps of the unit interval preserving an absolutely continuous probability measure the conditions hold, except at the indifferent fixed point $x = 0$. Recent work of Tanja Schindler [28, Theorem 1.8] on trimmed Birkhoff sums has shown that for Gibbs-Markov maps the limit infimum estimate (d) can be improved to $\liminf \frac{S_n (\log\log n)^{k-1+\varepsilon}}{n^k} = \infty$ for any $\varepsilon > 0$.

2.2 Non-integrable observables on a class of intermittent type maps.

A simple model of intermittency, a form of Manneville-Pommeau map, is the class of maps T_α introduced by Liverani, Saussol and Vaienti in [23]

$$T_\alpha(x) = \begin{cases} x + 2^\alpha x^{1+\alpha}, \ 0 \leq x \leq 1/2 \\ 2x - 1, \ 1/2 \leq x \leq 1 \end{cases} \qquad 0 \leq \alpha < 1. \tag{2.1}$$

The map T_α has a unique absolutely continuous probability measure μ_α if $0 \leq \alpha < 1$. We will only consider the case of a probability measure, rather than an infinite measure preserving system. The density $h_\alpha(x)$ is Lipschitz and strictly positive on any interval of form $[a, 1]$, $a > 0$ but is unbounded at $x = 0$, where $h_\alpha(x) \sim x^{-\alpha}$, $\alpha > 0$.

Kim [18, Proposition 4.1] has shown that if $q \neq 0$ then any nested sequence of balls about q has the SBC property. We will improve this result to obtain the quantitative (QSBC) property, and obtain the following result.

Theorem 2.3 *Suppose $(T_\alpha, [0, 1], \mu_\alpha)$ is a Liverani-Saussol-Vaienti map with $0 \leq \alpha < 1$. Let $q \in [0, 1]$ and $\varphi(x) = d(x, q)^{-k}$ with $k \geq 1$. Define $S_n = \sum_{j=1}^{n} \varphi \circ T_\alpha^j$. Then if $q \neq 0$, for μ_α a.e. x and any $\varepsilon > 0$*

$$\liminf_{n\to\infty} \frac{S_n}{n^k (e^{-(\log n)^{\frac{1}{2}+\varepsilon}})^k} = \infty$$

and

$$\limsup_{n\to\infty} \frac{S_n}{n^k [\log(n)]^{k+\varepsilon}} = 0$$

In particular

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = k$$

If $q = 0$ then for any $\varepsilon > 0$

$$\liminf_{n\to\infty} \frac{S_n}{n^{k+\alpha-\varepsilon}} = \infty$$

and

$$\limsup_{n\to\infty} \frac{S_n}{n^{k+\alpha+\varepsilon}} = 0$$

In particular

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = k + \alpha$$

Remark 2.4 For generalized Manneville-Pommeau maps Dedecker, Gouëzel and Merlevede [31] proved that a strong law of large numbers with good error rates can be obtained for a large class of unbounded, but integrable observables.

Proof of theorem.

We first consider the case $q \neq 0$ and recall a proposition from [13]. We will use it to improve the SBC property estimate of Kim [18, Proposition 4.1] to the QSBC property.

Proposition 2.5 *Let X be a compact interval and let $\mathcal{P}$ be a countable partition of X into subintervals. Suppose that $(T, X, \mu, \mathcal{P})$ is a Gibbs-Markov system. Let (B_n) be a sequence of intervals in X for which there exists $C > 0$ such that $\mu(B_j) \le C\mu(B_i)$ for all $j \ge i \ge 0$. If $\sum_{n=0}^{\infty} \mu(B_n) = \infty$, then denoting $E_n = \sum_{j=1}^{n} \mu(B_j)$ for any $\varepsilon > 0$,*

$$\sum_{j=1}^{n} 1_{B_j} \circ T^j(x) = E_n + O(E_n^{1/2+\varepsilon})$$

for μ a.e. $x \in X$.

A first return time Young Tower (F, ν, Δ) may be constructed for this class of intermittent maps with base $\Delta = [1/2, 1]$ [29]. Every point $q \neq 0$ has a unique representation in such a first return time Tower, in the sense that there is a unique t such that $F^{-t}(q) \in \Delta$. Hence Proposition 2.5 shows that if $q \neq 0$ and (B_j) is a sequence of nested sequence of balls based about q then

$$\sum_{j=1}^{n} 1_{B_j} \circ T_\alpha^j(x) = E_n + O(E_n^{1/2+\varepsilon})$$

for μ_α a.e. $x \in X$.

Hence by the proof of Theorem 2.1 for μ_α a.e. x

$$\liminf \frac{S_n}{(ne^{-(\log n)^{\frac{1}{2}+\varepsilon}})^k} = \infty$$

for any $\varepsilon > 0$, and as a consequence of Aaronson [1, Proposition 2.3.1] for any $\varepsilon > 0$

$$\limsup \frac{S_n}{n^k[\log(n)]^{k+\varepsilon}} = 0$$

Now we consider the case $q = 0$. For nested intervals based at $q = 0$ an interesting failure of the dynamical Borel-Cantelli lemma occurs, described in [18]. To understand this phenomenon let T_1 and T_2 be the two branches of the map T_α, with domains $[0, 1/2]$ and $[1/2, 1]$ respectively. Consider the sequence of balls $B_j = [0, \frac{1}{j^\gamma})$ for any $1 < \gamma \le \frac{1}{1-\alpha}$. Kim notes that $\sum_n \mu_\alpha(B_j)$ diverges (due to $h_\alpha(x) \sim x^{-\alpha}$) while $\sum_n m(B_j) < \infty$. Note that $T_1^{-1}(B_j) \subset B_j$. Hence the only way that $T_\alpha^j(x)$ can enter B_j for infinitely many j is that $T_\alpha^{j-1}(x) \in T_2^{-1}(B_j)$ for infinitely many j. However the density $h_\alpha(x)$ is strictly positive and Lipschitz on any interval $[a, 1]$ for $a > 0$ and so $\sum_j \mu_\alpha(T_2^{-1}(B_j)) \sim \sum_j m(T_2^{-1}(B_j)) < \infty$ and the sequence (B_j) is not Borel-Cantelli.

We now consider the case of $q=0$ and $\varphi(x)=d(x,0)^{-k}$. In this setting using Aaronson [1, Proposition 2.3.1] we solve $\int a(\varphi)\frac{1}{x^{\alpha}}dx<\infty$ which gives an upper bound roughly of form $\limsup \frac{S_n}{n^{k/(1-\alpha)}}=0$, which is not optimal (being too large as we will see).

To get a better estimate we will consider the dynamics near the indifferent fixed point. The following local analysis of a large class of Manneville-Pommeau maps (of which the Liverani-Saussol-Vaienti map is a subclass) is taken from [30]. Fix $\varepsilon_0>0$, let $x_0\in(0,\varepsilon_0]$ and define the sequence x_n by $x_{n-1}=T_\alpha x_n$. Young shows that $x_n\sim\frac{1}{n^{\beta}}$ where $\beta=\frac{1}{\alpha}$. In fact there is a uniform bound on the number of intervals $[\frac{1}{(m+1)^{\beta}},\frac{1}{m^{\beta}}]$ that meet each $[x_{n+1},x_n]$ and vice-versa.

This implies that if $x=\frac{1}{2}+\frac{1}{2m^{\gamma}}$ then $T_\alpha x=\frac{1}{m^{\gamma}}$. Writing $\frac{1}{m^{\gamma}}=x_n$ for some sequence as described above we have $\frac{1}{m^{\gamma}}=\frac{1}{n^{\beta}}$, hence it takes $n\sim m^{\gamma/\beta}=m^{\gamma\alpha}$ iterates j for $T_\alpha^{j+1}x$ to escape the region $[0,\varepsilon_0]$ i.e. $T_\alpha^{j+1}x<\varepsilon_0$ for $j<m^{\gamma\alpha}$. Note that $\sum_{j=1}^n\varphi(x_j)\geq\sum_{j=1}^n j^{k\beta}$ as $x_j\sim\frac{1}{j^{\beta}}$ and hence $S_n\geq n^{k\beta+1}$. Hence if

$$x=\frac{1}{2}+\frac{1}{2m^{\gamma}}$$

then

$$\sum_{j=1}^{n}\varphi\circ T_\alpha^j x\geq m^{(\gamma/\beta)(k\beta+1)}=m^{\gamma(k+\alpha)} \tag{*}$$

This gives a lower bound on $\liminf S_n$ since if we define $n_l(x)=\max\{1\leq j\leq n\}$ such that $T_\alpha^j(x)\in[1/2,1/2+\frac{1}{j}]$ then for any $\varepsilon>0$, $\liminf\frac{n_l}{n^{1-\varepsilon}}\geq 1$ by the arguments of the previous section (we use the weaker SBC estimate as the stronger QSBC estimate does not help in this argument). Furthermore once $T_\alpha^n x$ enters $[1/2,1/2+\frac{1}{n}]$ it spends $\sim n^{\alpha}$ iterates in the region $(0,\varepsilon_0)$ whence $S_{n+n^{\alpha}}\geq n^{k+\alpha-\varepsilon}$. As $\alpha<1$ this implies that $\liminf\frac{S_n}{n^{k+\alpha-\varepsilon}}=\infty$ for any $\varepsilon>0$.

We will now show $\limsup\frac{S_n}{n^{k+\alpha+\varepsilon}}=0$ for any ε, hence $\lim_{n\to\infty}\frac{\log S_n}{\log n}=k+\alpha$. We first sketch our argument. Let $0<\eta<1$ and $q\in(0,1]$. Then $\sum_{j=1}^n 1_{B(q,\frac{1}{j^{\eta}})}\circ T_\alpha^j(x)\sim n^{1-\eta}$ for μ_α a.e. x. Note that if $\delta>0$ then by Borel-Cantelli μ_α a.e. $x\in X$ has the property that $T_\alpha^n x\in B(q,(n\log^{1+\delta}n)^{-1})$ for only finitely many n. Asymptotically almost every x has the property that $T_\alpha^j x\in B(q,\frac{1}{j^{\eta}})$ for $\sim n^{1-\eta}$ iterates j in the interval $1\leq j\leq n$, after a certain $L(x)$, i.e. for $j\geq L(x)$, the maximum value that $\varphi\circ T_\alpha^{j+1}x$ attains if $T_\alpha^j(x)$ enters $B(q,\frac{1}{n^{\eta}})$ is $n^k\log^{k(1+\delta)}n$. We break up S_n for large n into the times j that $T_\alpha^j(x)$ enters $B(q,\frac{1}{n^{\eta}})$, roughly $n^{1-\eta}$ times where the value $\varphi\circ T_\alpha^{j+1}(x)$ is bounded by $n^k\log^{k(1+\delta)}n$ which thus contributes at most $n^{1-\eta}n^{k+\alpha}\log^{k(1+\delta)}n$ to S_n and the times j that $T_\alpha^j(x)$ enters $B^c(q,\frac{1}{n^{\eta}})$, which contributes at most $n.n^{\eta(k+\alpha)} = n^{1+\eta(k+\alpha)}$ to the sum S_n (using the estimate of line $(*)$). Incorporating the log term into the exponent, by choosing $\eta = \frac{k+\alpha}{k+\alpha+1}$ we obtain $\limsup S_n \leq n^{k+\frac{1}{k+1}+\alpha}$.

We will iterate this procedure. Choose $1 > \eta_1 > \eta_2 > \dots \eta_m > 0$ and for simplicity of notation let $B_{\eta_i} = B(q, \frac{1}{n^{\eta_i}})$.

The contribution of the iterates j that enter B_{η_1} we bound by the product of the maximum value they may attain, namely the value $n^{k+\alpha} \log^{k(1+\delta)} n$ and the number of times the point enters this sequence of sets $n^{1-\eta_1}$ to arrive at $n^{k+\alpha+1-\eta_1}$ (incorporating the log term into the exponent). This accounts for those iterates that enter $B_{\eta_1} \subset B_{\eta_2}$ and we bound the contribution of those that enter $B_{\eta_2} \setminus B_{\eta_1}$ by $n^{1-\eta_2}.n^{\eta_1(k+\alpha)} = n^{1-\eta_2+\eta_1(k+\alpha)}$. We bound the contribution of those that enter $B_{\eta_3} \setminus B_{\eta_2}$ by $n^{1-\eta_3}n^{\eta_2(k+\alpha)} = n^{1-\eta_3+\eta_2(k+\alpha)}$. Continuing in this way we have a sum of contributions of form $n^{1-\eta_{j+1}+\eta_j(k+\alpha)}$ terminating with the last contribution, those iterates j that lie in $B^c_{\eta_m}$ whose contribution we bound by $n^{\eta_m(k+\alpha)}.n = n^{1+\eta_m(k+\alpha)}$.

If $k \geq 1$, choosing $\varepsilon = \frac{1}{(k+\alpha)^m}$ and $\eta_i = 1-(k+\alpha)^{i-1}\varepsilon$ for $i = 1, \dots, m$ the leading term is $n^{k+\alpha+\varepsilon}$ corresponding to $n^{k+\alpha+1-\eta_1}$, thus $\limsup S_n \leq mn^{k+\alpha+\frac{1}{(k+\alpha)^m}}$ which implies the result as m was arbitrary. ∎

Liverani-Saussol-Vaienti Map.

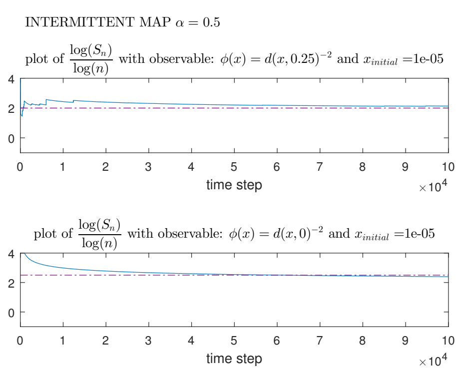

Figure: 1(a)

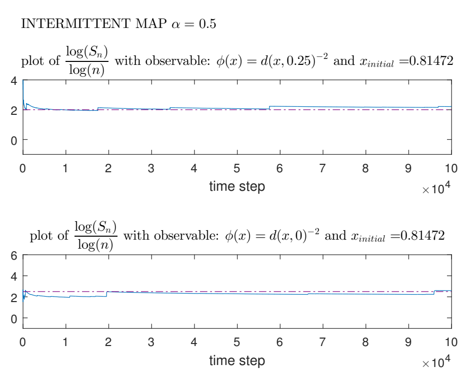

Figure: 1(b)

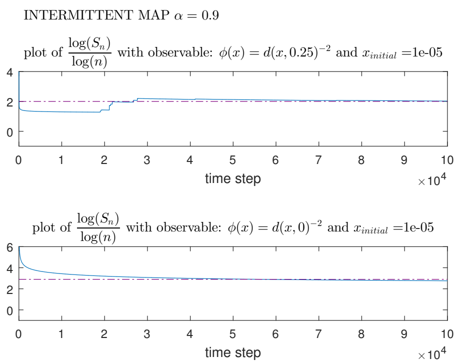

Figure: 1(c)

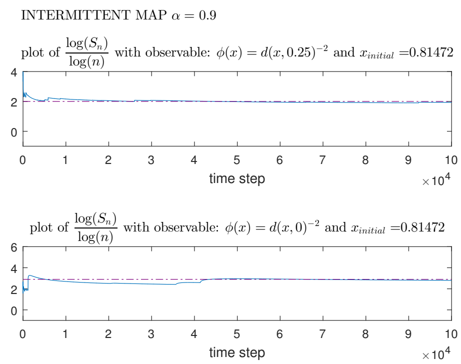

Figure: 1(d)

Predicted value of the limit is shown by a dotted line.

2.3 Dynamical systems with L^p, $(p > 1)$, densities and exponential decay of correlations.

In this section we consider dynamical systems with exponential decay of correlations, which possess absolutely continuous invariant measures (with respect to Lebesgue measure m) with densities $\frac{d\mu}{dm}$ in $L^p(m)$, $p > 1$.

Suppose (T, X, μ) is an ergodic measure preserving map of a probability space X which is a Riemannian manifold with Riemannian metric d. We assume:

(A) For all Lipschitz functions φ, ψ on X we have exponential decay of correlations in the sense that there exist constants $C, 0 < \theta < 1$ (independent of φ, ψ) such that

$$|E(\varphi\ \psi \circ T^k) - E(\varphi)E(\psi)| < C\theta^k \|\varphi\|_{\mathrm{Lip}} \|\psi\|_{\mathrm{Lip}}.$$

(B) There exist $r_0 > 0$, $0 < \delta < 1$ such that for $C > 0$, all $q \in X$ and all $0 < \varepsilon < r \leq r_0$

$$\mu\{\, x : r < d(x, q) < r + \varepsilon \,\} < C\varepsilon^{\delta}.$$

Under assumptions (A) and (B) Haydn, Nicol, Persson and Vaienti [14] showed:

Proposition 2.6 *Assume (T, X, μ) satisfies assumptions (A) and (B). Suppose $\mu(B_i) \geq C\frac{\log^{\beta} i}{i}$ for some $\beta > 0$, then if $E_n = \sum_{j=1}^{n} \mu(B_j)$ for μ a.e. $x \in X$.*

$$\sum_{j=1}^{n} 1_{B_j} \circ T^j(x) = E_n + O(E_n^{1/2+\varepsilon})$$

for any $\varepsilon > 0$.

In fact the density assumption $h := \frac{d\mu}{dm} \in L^p(m)$, $p > 1$ implies assumption (B).

Lemma 2.7 *Suppose* m *is Lebesgue measure on a* D*-dimensional manifold* X *and* $h := \frac{d\mu}{dm} \in L^p(m)$, $p > 1$*. Then for all* $0 < r < r_0$

$$\mu\{\, x : r < d(x,q) < r+\varepsilon \,\} < \varepsilon^{\delta}$$

for some $\delta > 0$

Proof of lemma: Let p be the conjugate of q, so that $\frac{1}{q} + \frac{1}{p} = 1$. Then $\int_{B_{r+\varepsilon}(q)/B_r(q)} d\mu = \int_{B_{r+\varepsilon}(q)/B_r(q)} h dx \le \|h\|_q m(x : r < d(x,q) < r+\varepsilon)^{\frac{1}{p}}$ which implies the result. ∎

In light of the above lemma we will often assume $\frac{d\mu}{dm} \in L^p(m)$, $p > 1$, rather than condition B (which is weaker but more awkward to state).

Remark 2.8 Any exponentially mixing volume preserving system satisfies (A) and (B), for example Sinai dispersing billiard maps with finite and infinite horizon [29, 5]. Furthermore for a volume preserving dynamical system the density $h(x) = \frac{d\mu}{dm}$ of the invariant measure is bounded above and is strictly positive. We consider the consequences of this in the next theorem.

Theorem 2.9 *Suppose a dynamical system* (T, X, μ) *satisfies (A) and* $q \in X$ *has density* $h = \frac{d\mu}{dm}$ *satisfying* $0 < C_1 < h(q) < C_2$ *for some constants* C_1, C_2*. Suppose also dim*$(X) = D$*. Then if* $\varphi(x) = d(x,q)^{-k}$, $k > D$,

$$\limsup_{n\to\infty} \frac{S_n}{n^{k/D}[\log(n)]^{\frac{k}{D}+\varepsilon}} = 0$$

and

$$\liminf_{n\to\infty} \frac{S_n}{n^{k/D}\big(e^{-[\log(n)]^{\frac{1}{2}+\varepsilon}}\big)^{\frac{k}{D}}} = \infty$$

for μ *a.e.* x *and any* $\varepsilon > 0$.

Remark 2.10 By ergodicity in the case $k = D$

$$\liminf_{n\to\infty} \frac{S_n}{n} = \infty$$

Proof.

Note that if $B(q,r)$ is a ball of small radius $r>0$ nested at q then $\mu(B(q,r)) \sim Cr^D$. Suppose $k>D$. Let $\varphi(x)=d(x,q)^{-k}$ and $a(x)=|x|^{\frac{D}{k}}/(|\log|x||)^{1+\eta}$. Then $\int a(\varphi(x))dx < \infty$. If we define $S_n=\sum_{j=1}^n \varphi\circ T^j$, then by [1, Proposition 2.3.1]

$$\frac{a(S_n)}{n} \to 0$$

for μ a.e. $x\in X$. Hence for any $\varepsilon>0$, for μ a.e. $x\in X$

$$\limsup \frac{S_n}{n^{k/D}[\log(n)]^{\frac{k}{D}+\varepsilon}} = 0$$

To obtain a limit infimum estimate we modify our previous argument. Let B_j be balls of μ (hence m) measure $\sim \frac{\log^\beta n}{n}$ nested about q. Let $E_n := \sum_{j=1}^n \mu(B_j)$

Define $n_l := \max\{0\le j\le n\}$ such that $T^j(x)\in B_j$ as before we have

$$\sum_{j=1}^{n} 1_{B_j}\circ T^j = E_n + O(E_n^{1/2+\delta})$$

$$\sum_{j=1}^{n_l} 1_{B_j}\circ T^j = E_{n_l} + O(E_{n_l}^{1/2+\delta})$$

By definition of n_l, $\sum_{j=1}^{n_l} 1_{B_j}\circ T^j = \sum_{j=1}^{n} 1_{B_j}\circ T^j$ and hence

$$E_n - E_{n_l} = O(E_n^{1/2+\delta})$$

We obtain

$$\log^{1+\beta} n - \log^{1+\beta} n_l = O(\log^{1/2+\gamma}(n))$$

where $\gamma=\delta+\frac{\beta}{2}$. As $x-y\le x^{1+\beta}-y^{1+\beta}$ for large y and $x>y$ we see that

$$n_l \ge ne^{-(\log n)^{\frac{1}{2}+\varepsilon}}$$

for any $\varepsilon>0$.

Note that balls of radius r based at q satisfy $\mu(B(q,r))\sim Cr^D$, and so we are able to bound S_n below by $M_{n_l} \ge (ne^{-[\log(n)]^{\frac{1}{2}+\varepsilon}})^{\frac{k}{D}}$.

Hence

$$\liminf \frac{S_n}{(ne^{-[\log(n)]^{\frac{1}{2}+\varepsilon}})^{\frac{k}{D}}} = \infty$$

for any $\varepsilon>0$. ∎

A recent result of J.Rivera-Letelier [22, Corollary B] states:

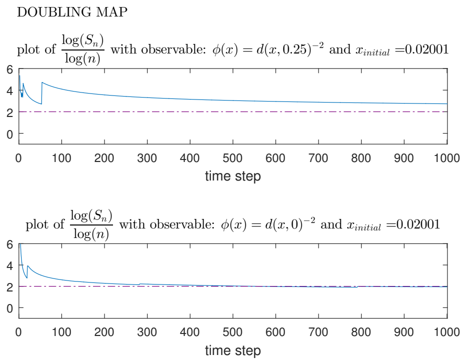

Figure: 2 Doubling Map.

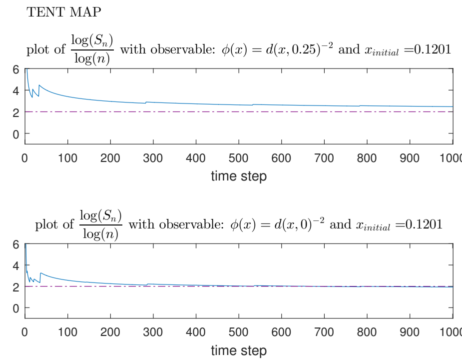

Figure: 3 Tent Map.

Predicted value of the convergence is marked in a dotted line.

Proposition 2.11 *Let T be a non-degenerate smooth interval map having an exponentially mixing absolutely continuous invariant probability measure μ. Then there is $p > 1$ such that the density h of μ with respect to Lebesgue measure m is in $L^p(m)$. Moreover, μ can be obtained through a Young tower with an exponential tail estimate.*

For such maps if the invariant density at q satisfies $h(x) \sim Cd(q,x)^{-\alpha}$, $\alpha > 0$, then we have the estimates:

Theorem 2.12 *Suppose a dynamical system (T, X, μ) satisfies (A) and $q \in X$ has density satisfying $h(x) \sim Cd(q,x)^{-\alpha}$, $\alpha > 0$ for x near q. Suppose also dim$(X) = D$. Then if $\varphi(x) = d(x,q)^{-k}$, $k \geq D - \alpha$,*

$$\limsup_{n\to\infty} \frac{S_n}{n^{k/(D-\alpha)}[\log(n)]^{k+\varepsilon}} = 0$$

and

$$\liminf_{n\to\infty} \frac{S_n}{n^{k/(D-\alpha)}\left(e^{-[\log(n)]^{\frac{1}{2}+\varepsilon}}\right)^{\frac{k}{D-\alpha}}} = \infty$$

for μ a.e. x any $\varepsilon > 0$. Hence

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = \frac{k}{D-\alpha}$$

Remark 2.13 This result contrasts with that of the intermittent map where at the indifferent fixed point $x = 0$, with density $h(x) \sim x^{-\alpha}$ it was shown that for the observable $\varphi(x) = x^{-k}$, $\lim_{n\to\infty} \frac{\log S_n}{\log n} = k + \alpha$.

Proof:

The proof is an obvious modification of the proof of the previous theorem. Let $\widetilde{D} = D - \alpha$ and define $a(x) = \frac{|x|^{\widetilde{D}/k}}{(\log|x|)^{1+\eta}}$. Then $\int a(\varphi(x))dx < \infty$ and by [1, Proposition 2.3.1] $\frac{a(S_n)}{n} \to 0$ and hence

$$\limsup \frac{S_n}{n^{k/(\widetilde{D})}[\log(n)]^{k+\varepsilon}} = 0$$

We now obtain our limit infimum estimate.

Let B_j be balls of μ measure $\sim \frac{\log^\beta n}{n}$ nested about q. Define $n_l := \max\{0 \le j \le n\}$ where $T^j(x) \in B_j$ as before we have

$$\sum_{j=1}^{n} 1_{B_j} \circ T^j = E_n + O(E_n^{1/2+\delta})$$

$$\sum_{j=1}^{n_l} 1_{B_j} \circ T^j = E_{n_l} + O(E_{n_l}^{1/2+\delta})$$

and hence

$$E_n - E_{n_l} = O(E_n^{1/2+\delta})$$

We have

$$\log^{1+\beta} n - \log^{1+\beta} n_l = O(\log^{1/2+\gamma}(n))$$

where $\gamma = \delta + \frac{\beta}{2}$. As $x - y \le x^{1+\beta} - y^{1+\beta}$ for large y large and $x > y$ we see that as in the previous theorem

$$n_l \ge n e^{-(\log n)^{\frac{1}{2}+\varepsilon}}$$

for any $\varepsilon > 0$.

Note that balls of radius r based at q satisfy $\mu(B(q,r)) \sim C r^{\widetilde{D}}$ we see that $S_n \ge M_{n_l}$ implies

$$\liminf \frac{S_n}{(n e^{-[\log(n)]^{\frac{1}{2}+\varepsilon}})^{\frac{k}{D-\alpha}}} = \infty$$

for any $\varepsilon > 0$. ∎

Corollary 2.14 *Suppose $T(x) = 4x(1-x)$ is a unimodal map of the interval $[0,1]$. Let $\varphi(x) = d(x,q)^{-k}$, then if $q = 0$ or $q = 1$*

$$\lim_{n\to\infty} \frac{\log(S_n)}{\log n} = 2k$$

while if $q \in (0,1)$

$$\lim_{n\to\infty}\frac{\log(S_n)}{\log n}=k$$

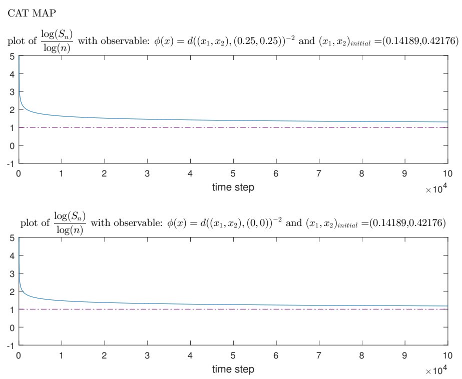

Figure: 4 Hyperbolic Toral Automorphism.

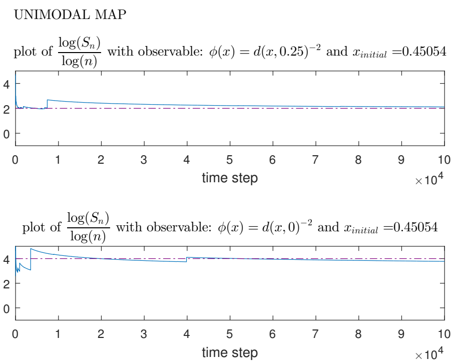

Figure: 5 Unimodal Map.

Predicted value of the convergence is marked in a dotted line.

Proof of corollary:

This map has invariant density $h(x)=\frac{1}{\sqrt{\pi x(1-x)}}$. First note that the unimodal map has density $h(x)\sim\frac{1}{\sqrt{x}}$ for $q=0$ and $q=1$ which implies the result. ∎

3 Conclusion.

Dynamical Borel Cantelli lemmas and a simple lemma of Aaronson [1, Proposition 2.3.1] give useful bounds on the rate of growth of positive non-integrable functions on ergodic dynamical systems. In the case of Gibbs-Markov maps the lower bounds we obtain are not optimal [28]. These results suggest that if the growth of the Birkhoff sum is determined mainly by the maximal term then Quantitive Borel-Cantelli estimates give found bounds for upper and lower rates of growth. In settings where there is an indifferent fixed point then the rate of growth of the Birkhoff sum of an observable maximized at that point is determined by intermediate values. It would be of interest to develop insights into a broader class of examples. It would also be interesting to explore more examples in the setting of Birkhoff sums of functions $\varphi = \varphi^+ - \varphi_-$ on ergodic probability measure preserving systems with non-integrable positive and non-integrable negative parts.

4 Appendix

We define the term Gibbs-Markov map, as used in this paper, in this section.

Let $(X, \mathcal{B}, m)$ be a Lebesgue probability space with $X \subset \mathbb{R}$. Let $\mathcal{P}$ be a countable measurable partition of X such that $m(\alpha) > 0$ for all $\alpha \in \mathcal{P}$.

A measure-preserving map $T : X \to X$ is said to be a **Markov map** if the following are satisfied.

1. $\mathcal{P}$ generates $\mathcal{B}$ under T
2. (Markov property) For all $\alpha, \beta \in \mathcal{P}$, if $m(T(\alpha) \cap \beta) > 0$ then $\beta \subset T(\alpha)$.
3. (local invertibility) For all $\alpha \in \mathcal{P}$, $T|\alpha$ is invertible.

For integer $n \geq 0$, let $\mathcal{P}_n$ be the partition of X defined by

$$\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{P})$$

Let $J_T = \frac{d(m \circ T)}{dm}$.

The quintet $(X, \mathcal{B}, m, T, \mathcal{P})$ is said to be a **Gibbs-Markov system** if T is a Markov map and the following properties also hold.

1. (full branches) For all $\alpha \in \mathcal{P}$, $T(\alpha) = X$, mod m.

2. (uniform expansion) There exists $K_1 > 0$ and $\gamma_1 \in (0,1)$ such that $m(\alpha) \leq K_1\gamma_1^n$ for all $n \geq 0$ and $\alpha \in \mathcal{P}_n$.

3. (distortion control) There exists $K_2 > 0$ and $\gamma_2 \in (0,1)$ such that for all $n \geq 0$ and $\alpha \in \mathcal{P}_n$, we have
$$|\log\left(\frac{J_{T^n}(x)}{J_{T^n}(y)}\right)| \leq K_2\gamma_2^n \tag{4.1}$$
for all $x, y \in \alpha$.

Remark 4.1 Some authors weaken the full-branch condition in the definition of Gibbs-Markov systems by requiring merely that $m(T(\alpha)) > K > 0$ for some K independent of α.

References

[1] J. Aaronson, On the ergodic theory of non-integrable functions and infinite measure spaces, *Israel J. Math.* **27** (1977), no. 2, 163-173.

[2] J. Aaronson, Z. Kosloff, B.Weiss. Symmetric Birkhoff sums in infinite ergodic theory, *preprint http://arxiv.org/abs/1307.7490.*

[3] J. Aaronson and H. Nakada, Trimmed sums for non-negative, mixing stationary processes, *Stochastic Process. Appl.* 104 (2003), no. 2, 173-192.

[4] J. Aaronson and H. Nakada, On the mixing coefficients of piecewise monotonic maps, Probability in mathematics. *Israel J. Math.* 148 (2005), 1-10.

[5] N. Chernov. Decay of correlations in dispersing billiards. *Journal of Statistical Physics*, **94**, (1999), 513-556.

[6] N. Chernov and D. Kleinbock. Dynamical Borel–Cantelli lemmas for Gibbs measures, *Israel J. Math.* **122** (2001), 1–27.

[7] Y. S. Chow and H. Robbins. On sums of independent random variables with infinite moments, and "fair"games, *Proc. Nat. Acad. Sci.* , **47**, (1961), 330-335.

[8] D. Dolgopyat. Limit theorems for partially hyperbolic systems, Trans. AMS **356** (2004), 1637–1689.

[9] W. Feller. A limit theorem for random variables with infinite moments. *American Journal of Mathematics*, **68**(2), (1946) 257-262.

[10] W. Feller. An Introduction to Probability Theory and Its Applications, Volume 1. Wiley, New York, 3rd edition, 1968.

[11] S. Galatolo and D. H. Kim, The dynamical Borel-Cantelli lemma and the waiting time problems. *Indag. Math. (N.S.)* 18 (2007), no. 3, 421-434.

[12] S. Gouëzel, A Borel–Cantelli lemma for intermittent interval maps, *Nonlinearity*, **20(6)** (2007), 1491–1497.

[13] C. Gupta, M. Nicol and W. Ott, A Borel–Cantelli lemma for non-uniformly expanding dynamical systems, *Nonlinearity* **23(8)** (2010), 1991–2008.

[14] N. Haydn, M. Nicol, T. Persson, S. Vaienti. A note on Borel-Cantelli lemmas for non-uniformly hyperbolic dynamical systems.*Ergodic Theory Dynam. Systems* **3**3 (2013), 2, 475-498.

[15] M. P. Holland, M. Nicol and A. Török. Almost sure convergence of maxima for chaotic dynamical systems.*Stochastic Process. Appl.* **126** (2016), 10, 3145–3170.

[16] H. Kesten and R. A. Maller. The effect of trimming on the strong law of large numbers. *Proceedings of the London Mathematical Society*, (3) 71 (1995), no. 2, 441–480.

[17] J. Jaerisch, M. Kesseböhmer, B. Stratmann. A Fréchet law and an Erdös-Philipp law for maximal cuspidal windings. *Ergodic Theory Dynam. Systems*, 33 (2013), no. 4, 1008–1028.

[18] D. Kim, The dynamical Borel–Cantelli lemma for interval maps, *Discrete Contin. Dyn. Syst.* **17** (2007), no. 4, 891–900.

[19] Z. Kosloff. A universal divergence rate for symmetric Birkhoff Sums in infinite ergodic theory, *arxiv preprint 1412.1242 v2.*

[20] N. Luzia, *Borel-Cantelli lemma and its applications*. Trans. Amer. Math. Soc. 366 (2014), no. 1, 547–560

[21] E. Lesigne and D. Volny. Large deviations for martingales, *Stochastic Processes and their Applications*, **96(1)**, (2001), 143-159.

[22] J. Rivera-Letelier. Asymptotic expansion of smooth interval maps. *arxiv:1204.3071v2*

[23] C. Liverani, B. Saussol, S. Vaienti, A probabilistic approach to intermittency, *Ergodic theory and dynamical systems*, **19** (1999), 671–685.

[24] F. Maucourant, *Dynamical Borel–Cantelli lemma for hyperbolic spaces*, Israel J. Math., **152** (2006), 143–155.

[25] I. Melbourne and M. Nicol. Large deviations for non-uniformly hyperbolic dynamics systems, *Transactions of the AMS* **360** (2008), 6661-6676.

[26] W. Phillipp, Some metrical theorems in number theory, *Pacific J. Math.* **20** (1967) 109–127.

[27] Vladimir G. Sprindzuk, *Metric theory of Diophantine approximations*, V. H. Winston and Sons, Washington, D.C., 1979, Translated from the Russian and edited by Richard A. Silverman, With a foreword by Donald J. Newman, Scripta Series in Mathematics. MR MR548467 (80k:10048).

[28] T. Schindler. Generalized strong laws of large numbers for intermediately trimmed sums for non-negative stationary processes. *Thesis, Mathematik and Informatik, Universităt Bremen*, 2015.

[29] L.-S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.* **147** (1998) 585–650.

[30] L.-S. Young. Recurrence times and rates of mixing. *Israel J. Math.* **110** (1999) 153–188.

[31] Jerome Dedecker, Sébastien Gouëzel and Florence Merlevede. Some almost sure results for unbounded functions of intermittent maps and their associated Markov chains, *Ann. Inst. Henri Poincaré Probab. Stat.* 46 (2010), no. 3, 796?821.