Erdös Rényi laws for dynamical systems

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Abstract

We establish Erdös-Rényi limit laws for Lipschitz observations on a class of non-uniformly expanding dynamical systems, including logistic-like maps. These limit laws give the maximal average of a time series over a time window of logarithmic length. We also give results on maximal averages of a time series arising from Hölder observations on intermittent-type maps over a time window of polynomial length. We consider the rate of convergence in the limit law for subshifts of finite type and establish a one-sided rate bound for Gibbs-Markov maps.

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1 Introduction

The Erdös-Rényi law was first formulated for independent and identically distributed random variables in 1970 ([11]) as follows:

Theorem 1.1 *Let $(X_n)_{n\geq 1}$ be an independent identically distributed (iid) sequence of non-degenerate random variables, and put $S_n = X_1 + ... + X_n$. Assume that the moment generating function $\varphi(t) = E(e^{tX_1})$ exists in some*

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interval U containing $t = 0$. For each $\alpha > 0$, define $\psi_\alpha(t) = \varphi(t)e^{-\alpha t}$. For those α for which ψ_α attains its minimum at a point $t_\alpha \in U$, set $c_\alpha = -1/\log \psi_\alpha(t_\alpha)$. Then

$$\lim_{N\to\infty} \max\{(S_{n+[c_\alpha \log N]} - S_n)/[c_\alpha \log N] : 1 \le n \le N - [c_\alpha \log N]\} = \alpha.$$

To appreciate the significance of the theorem we quote from [11]: Suppose X_i is an iid sequence taking on the values ± 1 with equal probability and define $\theta(N, K(N))$ by

$$\theta(N, K(N)) := \max_{0 \le n \le N-K(N)} \frac{S_{n+K(N)} - S_n}{K(N)}.$$

$\theta(N, K(N))$ may be interpreted as the maximal average gain over a time window of length $K(N)$ up to time N. A straightforward calculation using the strong law of large numbers shows that if $\lim_{N\to\infty} \frac{K(N)}{\log N} = \infty$ then $\lim_{N\to\infty} \theta(N, K(N)) = 0$, P a.s. However if $K(N) \le c \log_2 N$ with $0 < c < 1$ then in the limit of large N with probability one there is at least one $n < N - K(N)$ such that $X_{n+1} = X_{n+2} = \ldots = X_{n+K(N)} = 1$ so that $\lim_{N\to\infty} \theta(N, K(N)) = 1$ P a.s.

Thus in this setting of a fair game the Erdös-Rényi law gives information on the maximal average gain of a player in a fair game precisely in the case where the length of the time window ensures $\lim_{N\to\infty} \theta(N, K(N))$ has a non-degenerate limit. As another application, Erdös and Rényi take X_i to be iid with the standard normal distribution $N(0, 1)$ and give a simple proof of a remarkable result of Lévy [14]: if $B(t)$ is canonical Brownian motion then

$$\lim_{t\to 0} P(|B(t+h) - B(t)| < \lambda \sqrt{2h \log \frac{1}{h}} \text{ for } 0 \le t \le 1-h\ ) = \begin{cases} 1 & \text{if } \lambda > 1; \\ 0 & \text{if } \lambda < 1. \end{cases}$$

In this note we establish Erdös-Rényi limit laws for certain non-uniformly expanding maps. We also discuss stronger versions which give rates of convergence. In the dynamical systems context such a result has first been obtained by Grigull [13] in 1993, later by Chazottes and Collet [5] for uniformly expanding maps of the interval, and for Gibbs-Markov dynamics by Denker and Kabluchko [8]. The results of Chazottes and Collet [5] also give a convergence rate (as do Deheuvels et al [9] for the independent case). The convergence rates we give may not be optimal.

2 Erdös-Rényi law

Suppose that (T, X, μ) is a probability preserving transformation and $\varphi : X \to \mathbb{R}$ is a mean-zero integrable function i.e. $E(\varphi) := \int_X \varphi \, d\mu = 0$. Let $S_n(\varphi) := \varphi + \varphi \circ T + \ldots + \varphi \circ T^{n-1}$.

Definition 2.1 *A mean-zero integrable function $\varphi : X \to \mathbb{R}$ is said to satisfy a large deviation principle with rate function $I(\alpha)$, if there exists a neighborhood U of 0 and a strictly convex function $I : U \to \mathbb{R}$, non-negative and vanishing only at $\alpha = 0$, such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(S_n(\varphi) \geq n\alpha) \quad = \quad -I(\alpha) \tag{1}$$

for all $\alpha > 0$ in U and

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(S_n(\varphi) \leq n\alpha) \quad = \quad -I(\alpha) \tag{2}$$

for all $\alpha < 0$ in U.

The rate function $I(\alpha)$ is also called the information function. Throughout this paper we will concentrate on the case $\alpha > 0$ as the case $\alpha < 0$ is identical with the obvious modifications of statements.

The first result is well known and may be established by an adapted proof from Erdös and Rényi [11] (see also Grigull [13] or Denker and Kabluchko [8] where this method has been used). We give the proof for completeness. We let the Gauss bracket [.] denote the integer part of a number.

Proposition 2.2 *(a) Suppose that φ satisfies a large deviation principle with rate function I defined on the open set U. Let $\alpha > 0$ and set*

$$l_n = l_n(\alpha) = \left[\frac{\log n}{I(\alpha)}\right] \qquad n \in \mathbb{N}.$$

Then the upper Erdös-Rényi law holds, that is, for μ a.e. $x \in X$

$$\limsup_{n\to\infty} \max\{S_{l_n}(\varphi) \circ T^j(x)/l_n : 0 \leq j \leq n - l_n\} \leq \alpha.$$

(b) Moreover, if for some constant $C > 0$ and integer $\tau \geq 0$ for each interval A

$$\mu(\bigcap_{m=0}^{n-l_n} \{S_{l_n}(\varphi) \circ T^m \in A\}) \quad \leq \quad C[\mu(S_{l_n} \in A)]^{n/(l_n)^\tau} \tag{3}$$

then the lower Erdös-Rényi law holds as well, that is, for μ a.e. $x \in X$

$$\liminf_{n\to\infty} \max\{S_{l_n}(\varphi)\circ T^j/l_n : 0 \le j \le n-l_n\} \ge \alpha.$$

Remark 2.3 *Assumptions (a) and (b) of Proposition 2.2 together imply that*

$$\lim_{n\to\infty} \max_{0\le m\le n-l_n} \frac{S_{l_n}\circ T^m}{l_n} = \alpha.$$

Proof. Let $\alpha \in U$ and l_n be as above. We consider the case $\alpha > 0$. To simplify notation we write $S_n(\varphi)$ as S_n.

Choose $\epsilon > 0$ such that $\alpha+\epsilon \in U$, and define

$$A_n(\epsilon) = \{x \in X : \max_{0\le m\le n-l_n} S_{l_n}\circ T^m \ge (\alpha+\epsilon)l_n\}.$$

If $0 < 2\delta < I(\alpha+\epsilon) - I(\alpha)$ (I is strictly convex), then there exists $N \in \mathbb{N}$ such that for $n > N$

$$\begin{aligned}\mu(A_n(\epsilon)) &\le n\mu(S_{l_n} \ge (\alpha+\epsilon)l_n)\\ &\le ne^{-l_n(I(\alpha+\epsilon)-\delta)}\\ &\le ne^{-l_n(I(\alpha)+\delta)}\\ &\le n^{-\frac{\delta}{I(\alpha)}}.\end{aligned}$$

Apply this estimate for $n = k^p$ where $p > \frac{I(\alpha)}{\delta}$ is an integer to obtain via the Borel-Cantelli lemma that for μ a.e. $x \in X$

$$\limsup_{k\to\infty} \max_{0\le m\le k^d-l_{k^d}} S_{l_{k^d}}\circ T^m/l_{k^d} \le \alpha+\epsilon.$$

Replacing l_n by $l'_n = l_n - 1$ yields

$$\limsup_{k\to\infty} \max_{0\le m\le k^d-l'_{k^d}} S_{l_{k^d}}\circ T^m/l_{k^d} \le \alpha+\epsilon.$$

Now take any $n \in \mathbb{N}$. Choose k such that $(k-1)^d < n \le k^d$. Then, for k large, $l_{(k-1)^d}$ and l_{k^d} differ by at most one. Hence $l_n = l_{k^d}$ or $l_n = l'_{k^d}$ and therefore

$$S_{l_n}\circ T^m = S_{l_{k^d}} \quad \text{or} \quad S_{l_n}\circ T^m = S_{l'_{k^d}}\circ T^m$$

for $0 \le m \le n$. This shows that for μ a.e. $x \in X$

$$\limsup_{n\to\infty} \max_{0\le m\le n-l_n} S_{l_n}\circ T^m/l_n \le \alpha+\epsilon.$$

Letting $\epsilon \to 0$ proves the first part of the proposition.

For the converse inequality, let $\alpha > 0$ and choose $\epsilon > 0$ such that $\alpha - \epsilon > 0$. Define

$$B_n(\epsilon) = \{\max_{0 \le m \le n - l_n} S_{l_n} \circ T^m \le l_n(\alpha - \epsilon)\}.$$

Define also

$$C_m(\epsilon) = \{S_{l_n} \circ T^m \le l_n(\alpha - \epsilon)\}.$$

Then $B_n(\epsilon) = \bigcap_{m=0}^{n-l_n} C_m(\epsilon)$ and by assumption

$$\mu(B_n(\epsilon)) \le C\mu(S_{l_n} \le l_n(\alpha - \epsilon))^{n/(l_n)^\tau}.$$

Using the large deviation property for $C_0(\epsilon)^c := X \setminus C_0(\epsilon)$, for any $\delta_1 > 0$ for large n one obtains $\mu(C_0(\epsilon)^c) \ge e^{-l_n(I(\alpha-\epsilon)+\delta_1)} \ge e^{-(\frac{I(\alpha-\epsilon)}{I(\alpha)} \log n} e^{\delta_1 \frac{\log n}{I(\alpha)}}$. For large n (note δ_1 may be taken to approach 0 as $n \to \infty$) one obtains

$$1 - \mu(C_0(\epsilon)) \ge e^{-(1-\delta)\log n}$$

for some $0 < \delta < 1$. Therefore

$$\begin{aligned} \mu(B_n(\epsilon)) &\le C\left[1 - e^{-(1-\delta)\log n}\right]^{n/(l_n)^\tau} \\ &= O(\exp -n^{\delta'}) \end{aligned}$$

where δ' is any $0 < \delta' < \delta$. The lower bound follows from the Borel-Cantelli lemma.

Remark 2.4 *It is clear from the proof that to obtain the upper Erdös-Rényi law it suffices to have exponential large deviations given by a rate function, while for the lower Erdös-Rényi law it suffices to show that for every $\epsilon > 0$ the series $\sum_{n>0} \mu(B_n(\epsilon))$, where $B_n(\epsilon) = \{\max_{0 \le m \le n-l_n} S_{l_n} \circ T^m \le l_n(\alpha - \epsilon)\}$ is summable.*

3 One-dimensional non-uniformly expanding maps.

We now give some applications of Proposition 2.2. Melbourne and Nicol [19] have established the existence of a rate function $I(.)$ for a broad class of non-uniformly expanding maps and non-uniformly hyperbolic systems (see also [21] for related results). For these systems the upper Erdös-Rényi law of Proposition 2.2 immediately holds for processes generated by Lipschitz functions. In this section we verify the mixing condition (3) of Proposition 2.2 to establish the lower Erdös-Rényi law for Lipschitz functions for a class of non-uniformly expanding maps. Our results rely on the existence of an exponential rate function, exponential decay of correlations and a bounded derivative for T.

Theorem 3.1 *Suppose that (T, X, μ) is a C^2 non-uniformly expanding map of the interval X with an absolutely continuous invariant probability measure μ satisfying exponential decay of correlations of the form: for all $\varphi \in Lip$, $\psi \in L^\infty$ we have*

$$|\int \varphi\psi \circ T^j \, dm - \int \varphi \, dm \int \psi \, dm| \leq C\theta^j \|\varphi\|_{Lip} \|\psi\|_\infty$$

where $0 < \theta < 1$ and C is a constant independent of φ and ψ.

Then any Lipschitz observation $\varphi : X \to \mathbb{R}$ has exponential deviations with a rate function and the upper and lower Erdös-Rényi laws of Proposition 2.2 hold.

Remark 3.2 *(i) Theorem 3.1 applies to the class of logistic maps $T(x) = 1 - ax^2$ with a in the set of parameters which lead to an absolutely continuous invariant measure and exponential decay of correlations.*

(ii) Theorem 3.1 also applies to the class of non-uniformly expanding maps of the interval modeled by what is often called a Young tower (see [3]) with exponential decay of correlations, as considered by Collet [6].

Proof. Melbourne and Nicol [19, Theorem 2.1] have established the existence of a rate function $I(.)$ for the class of maps in Theorem 3.1. The existence of a rate function implies that the upper Erdös-Rényi law of Proposition 2.2 holds. We will show that the lower Erös-Rényi law of Proposition 2.2 also holds by showing that for every $\epsilon > 0$, $\sum_{n>0} \mu(B_n(\epsilon))$ is summable (see Remark 2.4). To simplify notation we write S_n for $S_n(\varphi)$.

Since T is C^2, $|DT| < L$ for some $L > 0$. For $s > 0$ define $A_n^s = \{S_{l_n} \leq l_n(\alpha - s)\}$. We fix $\epsilon > 0$ and consider $A_n^{\epsilon/2} = \{S_{l_n} \leq l_n(\alpha - \epsilon/2)\}$ and $A_n^\epsilon = \{S_{l_n} \leq l_n(\alpha - \epsilon)\}$.

Let $0 < \eta < 1$. If $|x-y| < L^{-(1+\eta)m}$ then $|T^m x - T^m y| < L^{-\eta m}$ from the mean-value theorem applied to T^m and the fact that $|DT^m| < L^m$. Thus if n is large then $x \in A_n^{\epsilon}$ and $|x-y| < L^{-(1+\eta)l_n}$ implies that $y \in A_n^{\epsilon/2}$.

We may approximate the indicator function $1_{A_n^{\epsilon}}$ of A_n^{ϵ} by a Lipschitz function φ_{ϵ} of Lipschitz norm at most $L^{(1+\eta)l_n}$ satisfying $1_{A_n^{\epsilon}} \leq \varphi_{\epsilon} \leq 1$ and $\mu(A_n^{\epsilon}) < \int \varphi_{\epsilon} d\mu < \mu(A_n^{\epsilon/2})$. To do this let $F := A_n^{\epsilon}$ and define $h_1(z) = 0$ and $h_2(z) = 1 - d(z,F)L^{-(1+\eta)l_n}$. The fact that h_2 is Lipschitz with Lipschitz constant $L^{(1+\eta)l_n}$ is straightforward (see for example Stein [22, Section 2.1]). Thus $\varphi_{\epsilon}(z) := \max\{0, (1-d(z,F)L^{-(1+\eta)l_n}\}$ is Lipschitz with Lipschitz constant bounded by $L^{(1+\eta)l_n}$ and support in $A_n^{\epsilon/2}$.

Define $C_m(\epsilon) = \{S_{l_n} \circ T^m \leq l_n(\alpha - \epsilon)\}$ and $B_n(\epsilon) = \bigcap_{m=0}^{n-l_n} C_m(\epsilon)$. We use a blocking argument to take advantage of decay of correlations and intercalate by blocks of length $(\log n)^{\tau}$, $\tau > 1$. We define

$$E_n^0(\epsilon) := \bigcap_{m=0}^{[(n-(\log n)^{\tau})/(\log n)^{\tau})]} C_{m[(\log n)^{\tau}]}(\epsilon)$$

and in general for $0 \leq j < [\frac{n}{(\log n)^{\tau}}]$

$$E_n^j(\epsilon) := \bigcap_{m=0}^{[(n-(j+1)(\log n)^{\tau})/(\log n)^{\tau})]} C_{m[(\log n)^{\tau}]}(\epsilon).$$

Note that $\mu(B_n(\epsilon)) \leq \mu(E_n^0(\epsilon))$. For each j, let $\psi_j = 1_{E_n^j(\epsilon)}$ denote the indicator function of $E_n^j(\epsilon)$.

By decay of correlations we have

$$\begin{aligned}
\mu(E_n^0(\epsilon)) &\leq \int \varphi_{\epsilon} \cdot \psi_1 \circ T^{[(\log n)^{\tau}]} d\mu \\
&\leq C\theta^{(\log n)^{\tau}} \|\varphi_{\epsilon}\|_{Lip} \|\psi_1\|_{\infty} + \int \varphi_{\epsilon}\, d\mu \int \psi_1\, d\mu \\
&\leq \int \varphi_{\epsilon}\, d\mu \int \psi_1\, d\mu + C\theta^{(\log n)^{\tau}} (L^{(1+\eta)l_n}).
\end{aligned}$$

Applying a Lipschitz approximation and decay of correlations again to $\int \psi_1\, d\mu$ we iterate and conclude

$$\mu(E_n^0(\epsilon)) \leq nC\theta^{(\log n)^{\tau}} L^{(1+\eta)l_n} + \mu(A_n^{\epsilon/2})^{n/(\log n)^{\tau}}.$$

The term $nC\theta^{(\log n)^\tau} L^{(1+\eta)l_n}$ is clearly summable since $\tau > 1$. The same argument given in the proof of Proposition 2.2 using large deviations shows that $\mu(A_n^{\epsilon/2})^{n/(\log n)^\tau}$ is summable as $\mu(A_n^{\epsilon/2}) \leq 1 - e^{\log n[\frac{I(\alpha-\epsilon)}{I(\alpha)}+\delta_1]}$ where $\delta_1 \to 0$ as $n \to \infty$ so that $\mu(A_n^{\epsilon/2}) \leq 1 - e^{(1-\rho)\log n}$ for some $0 < \rho < 1$. Since the right hand side is summable the Borel-Cantelli lemma implies the result by Remark 2.4.

4 Intermittent maps and polynomial decay of correlations

In this section we investigate the maximal average over time-windows of varying length in a class of one-dimensional maps, commonly used as models of intermittent behavior. For simplicity we consider a well-understood model of intermittency, the Liverani-Saussol-Vaienti (LSV) map T_γ, though our results extend to the usual intermittent maps of the interval $X = [0, 1]$ (mod 1) which are expanding, except for an indifferent fixed point at $x = 0$ and which have an absolutely continuous invariant probability measure μ, such as the Manneville-Pomeau map $T(x) = x + x^{1+\gamma}$ where $0 < \gamma < 1$.

The LSV (see [15]) map of $X = [0, 1]$ (mod 1) is defined by

$$Tx = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } 0 \leq x < \frac{1}{2}; \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

For $0 < \gamma < 1$, T_γ has an absolutely continuous invariant probability measure μ which possesses a density $h(x)$ such that $h(x) \sim x^{-\gamma}$ as $x \to 0$. We will use the fact that if a_n is the pre-image of $T^{-n}(1)$ which is closest to 0 then $a_n \sim n^{-\frac{1}{\gamma}}$. Accordingly given $\epsilon > 0$, for sufficiently large n if $x \in [0, \frac{1}{n}]$ then $T^j(x) \in [0, \epsilon]$ for $j = 0, \dots, f(n)$ iterations where $\frac{f(n)}{n^\gamma} \to 1$ as $n \to \infty$. We will use this observation in the proof of some of our results. It is known that T has polynomial large deviations of order $\beta := \frac{1}{\gamma} - 1$. In fact Melbourne and Nicol [19] have shown in the case $0 < \gamma < \frac{1}{2}$ that if $\phi : X \to \mathbb{R}$ is Lipschitz, $\int \phi d\mu = 0$ then for all $\delta > 0$ there is a constant $C_1(\delta) > 0$ such that

$$\mu(\{x \in X : |S_n| \geq n\epsilon\}) \leq \frac{C_1}{n^{\beta-\delta}}.$$

Furthermore if $\phi(0) > \epsilon > 0$ then for any $\delta > 0$ there exists a constant $C_2(\delta) > 0$ such that

$$\frac{C_2}{n^{\beta+\delta}} \leq (\mu\{x \in X : |S_n| \geq n\epsilon\}).$$

Melbourne [18] subsequently extended these results to the case $0 < \gamma < 1$. Pollicott and Sharp [20] have shown, in the case $\phi(0) = 0$, that there exists $\bar{\beta} > 0$ satisfying

$$\mu(\{x \in A : |S_n| \geq n\epsilon\}) = O(e^{-\bar{\beta}n}).$$

Theorem 4.1 *Suppose $\phi : X \to \mathbb{R}$ is Lipschitz, $\int_X \phi \, d\mu = 0$.*

(a) If $\phi(0) > 0$ then for any $\frac{\gamma}{1-\gamma} < \tau_1 < 1$

$$\lim_{n\to\infty} \max_{0\leq m\leq n-n^{\tau_1}} S_{n^{\tau_1}} \circ T^m / n^{\tau_1} = 0.$$

(b) If $\phi(0) > 0$ and $0 < \tau_2 < \gamma$ then

$$\liminf_{n\to\infty} \max_{0\leq m\leq n-n^{\tau_2}} S_{n^{\tau_2}} \circ T^m / n^{\tau_2} \geq \phi(0).$$

Moreover if $\phi(0) = \max_{x\in X} \phi(x) > 0$ then equality holds in the form

$$\lim_{n\to\infty} \max_{0\leq m\leq n-n^{\tau_2}} S_{n^{\tau_2}} \circ T^m / n^{\tau_2} = \phi(0).$$

(c) If $\phi(0) = 0$ then for any $\sigma > (\bar{\beta})^{-1}$

$$\lim_{n\to\infty} \max_{0\leq m\leq n-(\log n)^{\tau}} S_{(\log n)^{\tau}} \circ T^m / (\sigma \log n) = 0.$$

Proof.

We prove (a) first. Choose $0 < \tau_1 < 1$ such that $\tau_1\beta > 1$, that is $\tau_1 > \frac{\gamma}{1-\gamma}$. Let $l_n = n^{\tau_1}$ and $2\delta = \tau_1\beta - 1$.

Let $\epsilon > 0$ and define

$$A_n := \{x \in X : \max_{0\leq m\leq n-l_n} |S_{l_n} \circ T^m| \geq l_n\epsilon\}.$$

Then $\mu(A_n) \leq n\mu(S_{l_n} \geq \alpha l_n) \leq C_1(\delta) n n^{-\tau_1\beta+\delta} = C_1 n^{-\delta}$.

Choose a $p > \frac{1}{\delta}$ so that $\delta p > 1$. We will consider the subsequence $n = k^p$. Via the Borel-Cantelli lemma for μ a.e. $x \in X$

$$\limsup_{k\to\infty} \max_{0\leq m\leq k^p-l_{k^p}} |S_{l_{k^p}} \circ T^m| / l_{k^p} \leq \epsilon$$

since $\sum_k Ck^{-p\delta} < \infty$. Note that $k^p - (k-1)^p = O(k^{p-1})$ hence, as ϕ is bounded,

$$\frac{S_{l_{k^p}} \circ T^m}{l_{k^p}} = \frac{S_{l_{(k-1)^p}} \circ T^m}{l_{k^p}} + O(\frac{1}{k})$$

where the implied constant is uniform in $x \in X$. As $\lim_{k\to\infty} \frac{k^p}{(k-1)^p} = 1$

$$\lim_{k\to\infty} \frac{|S_{l_{k^p}}|}{l_{k^p}} = \lim_{k\to\infty} \frac{|S_{l_{(k-1)^p}}|}{l_{(k-1)^p}}.$$

Since any $n \in \mathbb{N}$ satisfies $(k-1)^p \le n \le k^p$ for some k, it follows that

$$\limsup_{n\to\infty} \max_{0\le m\le n-l_n} |S_{l_n} \circ T^m|/l_n \le \epsilon.$$

As ϵ was arbitrary this proves (a).

To prove (b) let $0 < \tau_2 < \gamma$ and redefine $l_n = n^{\tau_2}$. Let $\epsilon > 0$. Since ϕ is Lipschitz with Lipschitz constant K say, there exists a neighborhood of 0 of length $K^{-1}\epsilon$ on which $\phi > \phi(0) - \epsilon$. It is known by results of Gouëzel [12] that if (A_n) is a sequence of intervals in X such that $\sum_n Leb(A_n) = \infty$ then for μ a.e. $x \in X$, $T^n(x) \in A_n$ infinitely often (the Borel-Cantelli property). Accordingly $T^n(x) \in [0, \frac{1}{n})$ infinitely often for μ a.e. $x \in X$. If n is sufficiently large then $T^n(x) \in [0, \frac{1}{n})$ implies that $T^{n+k}(x) \in [0, K^{-1}\epsilon]$ for $0 < k < n^{\tau_2} - C$ where C is a fixed constant. Hence for μ a.e. $x \in X$

$$\liminf_{n\to\infty} \max_{0\le m\le n-l_n} S_{l_n} \circ T^m/l_n \ge \phi(0) - \epsilon.$$

As $\epsilon > 0$ was arbitrary this proves the first assertion in (b).

Moreover if $\phi(0) = \max_{x\in X} \phi(x)$ note that $S_{l_n} \circ T^m/l_n \le \phi(0)$. Thus

$$\limsup_{n\to\infty} \max_{0\le m\le n-l_n} S_{l_n} \circ T^m/l_n \le \phi(0).$$

This proves (b).

To prove (c) recall that Pollicott and Sharp [20] have shown

$$\mu(\{x \in A : |S_n| \ge n\epsilon\}) = O(e^{-\bar{\beta}n}).$$

Let $l_n = [\sigma \log(n)]$ and

$$A_n := \{x \in X : \max_{0\le m\le n-l_n} |S_{l_n} \circ T^m| \ge l_n\epsilon\}.$$

Then $\mu(A_n) \leq n\mu(S_{l_n} \geq \epsilon l_n) \leq Ce^{-\bar{\beta}l_n}$ and the Borel-Cantelli lemma implies that

$$\limsup_{n\to\infty} \max_{0\leq m\leq n-l_n} |S_{l_n} \circ T^m|/l_n \leq \epsilon.$$

As $\epsilon > 0$ was arbitrary this concludes the proof of (c).

Remark 4.2 *The results here should be compared to those for iid random variables $(X_i)_{i\geq 1}$ with $EX_1 = 0$. Let*

$$D_n(r_n) = \max_{0\leq k\leq n-r_n} \sum_{j=k+1}^{k+r_n} X_j.$$

Then we have laws which are independent (or almost independent) of the distribution, like

(a) *If $r_n = n$, then $\lim_{n\to\infty} \frac{1}{n} D_n(n) = 0$ a.s. (the law of large numbers).*

(b) *If $r_n = n$, then $\limsup_{n\to\infty} \frac{D_n(n)}{\sqrt{2n\log\log n}} = EX_1^2$ a.s. (the law of iterated logarithm).*

(c) *If $\frac{n}{r_n} \to \infty$, under appropriate assumptions, the Czörgö-Révész law holds ([7]): $\limsup_{n\to\infty} \frac{D_n(r_n)}{b_n} = 1$ a.s. where $b_n = \sqrt{2r_n\left(\log\frac{n}{r_n}\right) + \log\log n}$.*

(d) *Under appropriate assumptions, $\lim_{n\to infty} \frac{D_n(r_n)}{r_n^{1/p}} = 1$ a.s. where $a > 0$, $1 < p < 2$ and $r_n = \left[\left(\frac{2}{a^2}\log n\right)^{p/(2-p)}\right]$ (Bock's theorem [4]).*

and also laws which depend on the distribution, like

(e) *If $r_n = [c\log n]$ then $\lim_{n\to\infty} \frac{D_n(r_n)}{r_n} = \alpha$, where $I(\alpha)c = 1$ (Erdös-Rényi law).*

(f) *If $c(n) = \frac{\log n}{r_n} \to \infty$ there are $\gamma(c(n))$ (depending on the distribution) such that $\lim_{n\to\infty} \frac{D_n(r_n)}{r_n\gamma(c(n))} = 1$ a.s. (Mason's law [17]).*

5 Rates of Convergence

In this section we refine our results to consider the rate of convergence in the Erdös-Rényi limit law. In order to obtain rates in this we need stronger assumptions, which will be verified for several examples.

In the setting of topologically mixing piecewise C^2 expanding maps of the interval (T, I, μ) Chazottes and Collet [5, Appendix A] have shown:

If $\varphi : I \to \mathbb{R}$ is of bounded variation then

$$\limsup_{n\to\infty} \max_{0\le m\le n-l_n} \frac{S_{l_n}\circ T^m - l_n\alpha}{\log l_n} \le \frac{1}{2I'(\alpha)} \quad a.e.$$

and

$$\liminf_{n\to\infty} \max_{0\le m\le n-l_n} \frac{S_{l_n}\circ T^m - l_n\alpha}{\log l_n} \ge -\frac{1}{2I'(\alpha)} \quad a.e.$$

To prove this they use the following result (see [5, Appendix A]), once again in the setting of topologically mixing piecewise C^2 expanding maps of the interval (T, X, μ). Such maps have exponential large deviations estimates given by a rate function, $I(.)$ and moreover: there is a compact interval K of the origin and constants $0 < c_1 < c_2$ such that for any $\alpha \in K/\{0\}$ and integer $n \ge 1 + I^{'}(\alpha)^{-4}$,

$$\frac{c_1}{I^{'}(\alpha)\sqrt{n}} e^{-nI(\alpha)} \le \mu(S_n \ge n\alpha) \le \frac{c_2}{I^{'}(\alpha)\sqrt{n}} e^{-nI(\alpha)}$$

The result of Chazottes and Collet is in agreement with the case of i.i.d random variables [9]. In the general context considered here we can prove a weaker result, the upper bound being the same, the lower bound depending on the choice of τ, or more precisely an appropriate Borel-Cantelli argument. We now state our rate theorem which adds to Proposition 2.2.

Theorem 5.1 *Suppose that φ satisfies a large deviation principle with a twice continuously differentiable rate function I defined on the open set $U \subset (0, \infty)$. Let $\alpha \in U$ and set*

$$l_n = l_n(\alpha) = \left[\frac{\log n}{I(\alpha)}\right] \qquad n \in \mathbb{N}.$$

(a) If there exists a $\overline{\kappa}$ such that

$$\limsup_{n\to\infty} \sup_{\alpha'\in U} \frac{1}{\log n} \left([\log \mu(S_n(\varphi) \ge n\alpha')] + nI(\alpha')\right) \le \overline{\kappa} \tag{4}$$

then

$$\limsup_{n\to\infty} \max_{0\le m\le n-l_n} \frac{S_{l_n}(\varphi)\circ T^m - l_n\alpha}{\log l_n} \le \frac{1+\overline{\kappa}}{I'(\alpha)} \quad a.e.$$

(b) If for some constant $C>0$ and integer $\tau\ge 0$ for each interval A

$$\mu(\bigcap_{m=0}^{n-l_n}\{S_{l_n}(\varphi)\circ T^m \in A\}) \le C[\mu(S_{l_n}\in A)]^{n/(l_n)^\tau}$$

and if there exists a $\underline{\kappa}$ such that

$$\liminf_{n\to\infty} \inf_{\alpha'\in U} \frac{1}{\log n}\left([\log\mu(S_n(\varphi)\ge n\alpha')] + nI(\alpha')\right) \ge \underline{\kappa} \tag{5}$$

then

$$\liminf_{n\to\infty} \max_{0\le m\le n-l_n} \frac{S_{l_n}(\varphi)\circ T^m - l_n\alpha}{\log l_n} \ge -\frac{\tau-\underline{\kappa}}{I'(\alpha)} \quad a.e.$$

Proof. (a) Let $d>\frac{1+\overline{\kappa}}{I'(\alpha)}$ and choose $\delta>0$ with $dI'(\alpha)-\overline{\kappa}-\delta>1$ and $B(\alpha,\delta)\subset U$. We have for all n large enough by the large deviation estimate (4)

$$\begin{aligned}
&\mu\left(\max_{0\le m\le n-l_n} S_{l_n}\circ T^m - l_n\alpha \ge d\log l_n\right)\\
\le\ & n\mu\left(S_{l_n}\ge l_n\left(\alpha + d\frac{\log l_n}{l_n}\right)\right)\\
\le\ & n l_n^{\overline{\kappa}+\delta}\exp\left[-l_n\left(I\left(\alpha+d\frac{\log l_n}{l_n}\right)\right)\right]\\
\le\ & n l_n^{\overline{\kappa}+\delta}\exp\left[-l_nI(\alpha) - I'(\alpha)d\log l_n + \frac{1}{2}I''(\zeta_n)d^2\frac{(\log l_n)^2}{l_n}\right]\\
=\ & O\left(l_n^{-I'(\alpha)d+\overline{\kappa}+\delta}\right),
\end{aligned}$$

where $\zeta_n\in B(\alpha,\delta)$.

Since $I'(\alpha)d-\overline{\kappa}-\delta>1$, it follows from the Borel-Cantelli Lemma, applied to the subsequence $q_k=[q^k]$ $(k\ge 1)$ where $q>1$, that

$$\limsup_{k\to\infty} \max_{0\le m\le q_k-l_{q_k}} (S_{l_{q_k}}\circ T^m - l_{q_k}\alpha)/\log l_{q_k} \le d. \tag{6}$$

We now proceed as in the proof of Proposition 2.2. Note that $l_{q^{k+1}} - l_{q^k}$ is bounded by a constant $\leq \log q + 1$. Therefore we can use the equation (6) replacing l_{q^k} by $l_{q^k} - j$ for $j = 0, 1, 2, ..., [1 + \log q]$. It follows that

$$\limsup_{n\to\infty} \max_{0\leq m\leq n-l_n} (S_{l_n} \circ T^m - l_n\alpha)/\log l_n \leq d.$$

(b) Let $d > \frac{\tau-\underline{\kappa}}{I'(\alpha)}$ and choose $\delta > 0$ with $dI'(\alpha) > \tau - \underline{\kappa} + \delta$ and $B(\alpha, \delta) \subset U$. Define

$$B_n = \{\max_{0\leq m\leq n-l_n} S_{l_n} \circ T^m \leq l_n\alpha - d\log l_n\},$$

and

$$C_m = \{S_{l_n} \circ T^m \leq l_n\alpha - d\log l_n\} \qquad m = 0, ..., n - l_n.$$

Then $B_n = \bigcap_{m=0}^{n-l_n} C_m$ and by assumption

$$\mu(B_n) \leq C[\mu(S_{l_n} \leq l_n\alpha - d\log l_n)]^{n/(l_n)^\tau}.$$

By the large deviation rate estimate (5) for all n sufficiently large,

$$\begin{aligned}
&\mu(S_{l_n} \geq l_n\alpha - d\log l_n) \\
&\quad\geq l_n^{\underline{\kappa}-\delta} \exp\left[-l_n I\left(\alpha - d\frac{\log l_n}{l_n}\right)\right] \\
&\quad= l_n^{\underline{\kappa}-\delta} \exp\left[-l_n\left(I(\alpha) - I'(\alpha)d\frac{\log l_n}{l_n} + \frac{1}{2}I''(\zeta_n)d^2\frac{(\log l_n)^2}{l_n^2}\right)\right] \\
&\quad\geq e^{-\left[\frac{\log n}{I(\alpha}\right]I(\alpha)} l_n^{\underline{\kappa}-\delta} \exp[I'(\alpha)d\log l_n - \frac{d^2}{2}I''(\zeta_n)\frac{\log l_n}{l_n}\log l_n] \\
&\quad= (1 + o(1))n^{-1}l_n^{I'(\alpha)d+\underline{\kappa}-\delta}
\end{aligned}$$

where $\zeta_n \in B(\alpha, \delta)$.

It follows that

$$\begin{aligned}
\mu(B_n) &\leq C\left(1 - (1 + o(1))n^{-1}l_n^{I'(\alpha)d+\underline{\kappa}-\delta}\right)^{n/(l_n)^\tau} \\
&= O\left(e^{-l_n^{I'(\alpha)d+\underline{\kappa}-\tau-\delta}}\right).
\end{aligned}$$

Since $d > \frac{\tau+\delta-\underline{\kappa}}{I'(\alpha)}$ then $\sum_k \mu(B_{[q^k]}) < \infty$ where $q > 1$ and by the Borel Cantelli lemma we arrive at

$$\liminf \max_{0\leq m\leq n-l_n} (S_{l_n} \circ T^m - l_n\alpha)/\log l_n \geq -d.$$

as before.

Example 5.2 *Let X_n $(n \geq 1)$ be an independent, identically distributed sequence of random variables with positive variance. Let $U \subset \mathbb{R}$ be the interval for which $c(t) = \log Ee^{tX_1} < \infty$. Let $\alpha \in \mathbb{R}$ and $t_\alpha \in U$ such that $I(\alpha) = e^{t_\alpha \alpha - c(t_\alpha)}$. Theorem 1 in [2] says that*

$$P(X_1 + ... + X_n \geq n\alpha) = \frac{1}{\sqrt{2\pi n}} e^{-nI(\alpha)} b_n (1 + o(1))$$

where $0 < \inf_n b_n < \sup_n b_n < \infty$ are non-random constants. This shows that $\overline{\kappa} = \underline{\kappa} = -\frac{1}{2}$, and we conclude that

$$\limsup_{n\to\infty} \max_{0\leq m\leq n-l_n} \frac{X_{m+1} + ... + X_{m+l_n} - l_n\alpha}{\log l_n} \leq \frac{1}{2I'(\alpha)} \quad a.e.$$

and

$$\liminf_{n\to\infty} \max_{0\leq m\leq n-l_n} \frac{X_{m+1} + ... + X_{m+l_n} - l_n\alpha}{\log l_n} \geq -\frac{3}{2I'(\alpha)} \quad a.e.$$

For the last statement we use the independence of the variables X_i to obtain trivially $\tau = 1$. Note that [9] improves this and also shows that the bounds are exact.

We are not able to improve our rate estimate for the general stationary case. Therefore, this example shows what can be expected in the best case.

Example 5.3 *let (T, X, μ) be a Gibbs measure on a subshift of finite type (Ω, T). It has been shown by Kesseböhmer (Korollar 5.3 in [16]) that for a Hölder–continuous function φ*

$$\mu(S_n(\varphi) \geq n\alpha) = A_n \frac{1}{n^{1/2}} e^{-nI(\alpha)} \tag{7}$$

uniformly in α, where the sequence A_n is bounded away from 0 and ∞. Therefore, for some neighborhood U of $\alpha > 0$

$$\lim_{n\to\infty} \sup_{\alpha'\in U} \frac{1}{\log n} [[\mu(S_n(\varphi) \geq n\alpha')] + nI(\alpha')] = -\frac{1}{2}.$$

Since $\tau = 1$ for the process $\varphi \circ T^k$, we obtain that

$$\limsup_{n\to\infty} \max_{0\leq m\leq n-l_n} \frac{S_{l_n}(\varphi) \circ T^m - l_n\alpha}{\log l_n} \leq \frac{1}{2I'(\alpha)} \qquad a.s.$$

and

$$\liminf_{n\to\infty} \max_{0\le m\le n-l_n} \frac{S_{l_n}(\varphi)\circ T^m - l_n\alpha}{\log l_n} \ge -\frac{3}{2I'(\alpha)} \qquad a.s.$$

This example shows in which way large deviation results in dynamics should be strengthened.

Example 5.4 *Let* (T, X, μ, γ) *be a mixing Gibbs-Markov map as defined in [1], where* γ *denotes the generating partition. It is known that any function* $\phi \in C(X)$ *which is constant on some refinement* $\gamma \vee T^{-1}\gamma \vee \ldots \vee T^{-p+1}\gamma$ *(where* $p \in \mathbb{N}$*) viewed as a function on* $\alpha^{\mathbb{N}}$ *generates a* ψ*-mixing process (see [8]).*

Let φ *be such a function and assume that* $\int \phi \, d\mu = 0$*. In this setting it is known that the free energy function*

$$c(t) = \lim_{n\to\infty} \frac{1}{n} \log E\left(e^{tS_n(\varphi)}\right)$$

exists and is twice continuously differentiable on a neighborhood U *of* 0 *and that the large deviation property holds uniformly in neighborhoods of* α *for each* $\alpha \in U$*, furthermore* τ *as defined in Proposition 2.2 may be taken as* $\tau = 1$*. Moreover, the information function is given by the Legendre-Fenchel transform*

$$I(\alpha) = \sup_{t\in\mathbb{R}} (t\alpha - c(t)).$$

We claim here that our theorem shows: If $\alpha \in U$ *then*

$$\limsup_{n\to\infty} \max_{0\le m\le n-l_n} \frac{S_{l_n}(\varphi)\circ T^m - l_n\alpha}{\log l_n} \le \frac{1}{I'(\alpha)} \quad a.e.$$

We prove this claim.

Let $m, r, k, q \in \mathbb{N}$ *satisfy*

$$m = kr + (k-1)q, \quad \text{and} \quad 0 \le \psi(q) < 1,$$

where $\psi(\cdot)$ *denotes the* ψ*-mixing coefficients. Since* γ *is a finite partition,* $\|\varphi\|_\infty < \infty$ *and there are functions* $m_i(t)$*,* $i = 1, 2$*, such that*

$$m_2(t) \le e^{t\varphi\circ T^k(x)} \le m_1(t) \qquad \forall x \in X.$$

Then, by ψ*-mixing,*

$$m_2(t)^{(k-1)q}(1-\psi(q))^{k-1}(Ee^{tS_r(\varphi)})^k \le Ee^{tS_m(\varphi)} \le m_1(t)^{(k-1)q}(1+\psi(q))^{k-1}(Ee^{tS_r(\varphi)})^k$$

Since $k-1=\frac{m-r}{r+q}$ *it follows that*

$$\begin{aligned}&m_2(t)^{\frac{q(1-r/m)}{r+q}}(1-\psi(q))^{\frac{1-r/m}{r+q}}\left(Ee^{tS_r(\varphi)}\right)^{\frac{1-q/m}{r+q}} \le \left(Ee^{tS_m(\varphi)}\right)^{\frac{1}{m}}\\ &\le m_1(t)^{\frac{q(1-r/m)}{r+q}}(1+\psi(q))^{\frac{1-r/m}{r+q}}\left(Ee^{tS_r(\varphi)}\right)^{\frac{1-q/m}{r+q}}.\end{aligned}$$

Taking the limit as $m\to\infty$ *while* q *and* r *are fixed, this shows that for any* $t\ge 0$

$$m_2(t)^{\frac{q}{r+q}}(1-\psi(q))^{\frac{1}{r+q}}\left(Ee^{tS_r(\varphi)}\right)^{\frac{1}{r+q}} \le e^{c(t)} \le m_1(t)^{\frac{q}{r+q}}(1+\psi(q))^{\frac{1}{r+q}}\left(Ee^{tS_r(\varphi)}\right)^{\frac{1}{r+q}}. \tag{8}$$

The theorem follows from Theorem 5.1 once it is shown that $\overline{\kappa}=0$.

Note that for $\alpha\in U$ *and any* $t>0$ *by (8) and setting* $r=n$

$$\begin{aligned}\mu(S_n(\varphi)\ge n\alpha)e^{nt\alpha-nc(t)} &\le e^{-nt\alpha}Ee^{tS_n(\varphi)}e^{nt\alpha}e^{-nc(t)}\\ &\le m_2(t)^{-q\frac{n}{n+q}}(1-\psi(q))^{-\frac{n}{n+q}}(Ee^{tS_n(\varphi)})^{\frac{q}{n+q}}.\end{aligned}$$

Taking the supremum over $t>0$ *over the left hand side shows that*

$$\mu(S_n(\varphi)\ge n\alpha)e^{nI(\alpha)} \le m_2(t_\alpha)^{-q\frac{n}{n+q}}(1-\psi(q))^{-\frac{n}{n+q}}\left(Ee^{t_\alpha S_n(\varphi)}\right)^{\frac{q}{n+q}},$$

where the supremum is attained at t_α *and is independent of* n. *Taking logarithms and dividing by* $\log n$ *we conclude that* $\overline{\kappa}\le 0$.

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