

Smooth Livšic regularity for piecewise expanding maps

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July 22, 2010

Abstract

We consider the regularity of measurable solutions χ to the cohomological equation

$$\phi = \chi \circ T - \chi,$$

where (T, X, μ) is a dynamical system and $\phi: X \rightarrow \mathbb{R}$ is a C^k valued cocycle in the setting in which $T: X \rightarrow X$ is a piecewise C^k Gibbs–Markov map, an affine β -transformation of the unit interval or more generally a piecewise C^k uniformly expanding map of an interval. We show that under mild assumptions, bounded solutions χ possess C^k versions. In particular we show that if (T, X, μ) is a β -transformation then χ has a C^k version, thus improving a result of Pollicott et al. [23].

1 Introduction

In this note we consider the regularity of solutions χ to the cohomological equation

$$\phi = \chi \circ T - \chi \tag{1}$$

where (T, X, μ) is a dynamical system and $\phi: X \rightarrow \mathbb{R}$ is a C^k valued cocycle. In particular we are interested in the setting in which $T: X \rightarrow X$ is a piecewise C^k Gibbs–Markov map, an affine β -transformation of the unit interval or more generally a piecewise C^k uniformly expanding map of an interval. Rigidity in this context means that a solution χ with a certain degree of

*M. N. and T. P. would like to thank the support and hospitality of Institute Mittag-Leffler, Djursholm, Sweden. The authors would also like to thank Viviane Baladi for providing some comments and references. Tomas Persson was supported by EC FP6 Marie Curie ToK programme CODY.

regularity is forced by the dynamics to have a higher degree of regularity. Cohomological equations arise frequently in ergodic theory and dynamics and, for example, determine whether observations ϕ have positive variance in the central limit theorem and have implication for other distributional limits (for examples see [20, 2]). Related cohomological equations to Equation (1) decide on stable ergodicity and weak-mixing of compact group extensions of hyperbolic systems [11, 20, 19] and also play a role in determining whether two dynamical systems are (Hölder, smoothly) conjugate to each other.

Livšic [13, 14] gave seminal results on the regularity of measurable solutions to cohomological equations for Abelian group extensions of Anosov systems with an absolutely continuous invariant measure. Theorems which establish that a priori measurable solutions to cohomological equations must have a higher degree of regularity are often called measurable Livšic theorems in honor of his work.

We say that $\chi: X \rightarrow \mathbb{R}$ has a C^k version (with respect to μ) if there exists a C^k function $h: X \rightarrow \mathbb{R}$ such that $h(x) = \chi(x)$ for μ a.e. $x \in X$.

Pollicott and Yuri [23] prove Livšic theorems for Hölder \mathbb{R} -extensions of β -transformations ($T: [0, 1) \rightarrow [0, 1)$, $T(x) = \beta x \pmod{1}$ where $\beta > 1$) via transfer operator techniques. They show that any essentially bounded measurable solution χ to Equation (1) is of bounded variation on $[0, 1 - \epsilon)$ for any $\epsilon > 0$. In this paper we improve this result to show that measurable coboundaries χ for C^k \mathbb{R} -valued cocycles ϕ over β -transformations have C^k versions (see Theorem 2).

Jenkinson [10] proves that integrable measurable coboundaries χ for \mathbb{R} -valued smooth cocycles ϕ (i.e. again solutions to $\phi = \chi \circ T - \chi$) over smooth expanding Markov maps T of S^1 have versions which are smooth on each partition element.

Nicol and Scott [15] have obtained measurable Livšic theorems for certain discontinuous hyperbolic systems, including β -transformations, Markov maps, mixing Lasota–Yorke maps, a simple class of toral-linked twist map and Sinai dispersing billiards. They show that a measurable solution χ to Equation (1) has a Lipschitz version for β -transformations and a simple class of toral-linked twist map. For mixing Lasota–Yorke maps and Sinai dispersing billiards they show that such a χ is Lipschitz on an open set. There is an error in [15, Theorem 1] in the setting of C^2 Markov maps — they only prove measurable solutions χ to Equation (1) are Lipschitz on each element $T\alpha$, $\alpha \in \mathcal{P}$, where \mathcal{P} is the defining partition for the Markov map, and not that the solutions are Lipschitz on X , as Theorem 1 erroneously states. The error arose in the following way: if χ is Lipschitz on $\alpha \in \mathcal{P}$ it is possible to extend χ as a Lipschitz function to $T\alpha$ by defining $\chi(Tx) = \phi(x) + \chi(x)$,

however extending χ as a Lipschitz function from α to $T^2\alpha$ via the relation $\chi(T^2x) = \phi(Tx) + \chi(Tx)$ may not be possible, as $\phi \circ T$ may have discontinuities on $T\alpha$. In this paper we give an example, (see Section 3), which shows that for Markov maps this result cannot be improved on.

Gouëzel [7] has obtained similar results to Nicol and Scott [15] for cocycles into Abelian groups over one-dimensional Gibbs–Markov systems. In the setting of Gibbs–Markov system with countable partition he proves any measurable solution χ to Equation (1) is Lipschitz on each element $T\alpha$, $\alpha \in \mathcal{P}$, where \mathcal{P} is the defining partition for the Gibbs–Markov map.

In related work, Aaronson and Denker [1, Corollary 2.3] have shown that if (T, X, μ, \mathcal{P}) is a mixing Gibbs–Markov map with countable Markov partition \mathcal{P} preserving a probability measure μ and $\phi: X \rightarrow \mathbb{R}^d$ is Lipschitz (with respect to a metric ρ on X derived from the symbolic dynamics) then any measurable solution $\chi: X \rightarrow \mathbb{R}^d$ to $\phi = \chi \circ T - \chi$ has a version $\tilde{\chi}$ which is Lipschitz continuous, i.e. there exists $C > 0$ such that $d(\tilde{\chi}(x), \tilde{\chi}(y)) \leq C\rho(x, y)$ for all $x, y \in T(\alpha)$ and each $\alpha \in \mathcal{P}$.

Bruin et al. [4] prove measurable Livšic theorems for dynamical systems modelled by Young towers and Hofbauer towers. Their regularity results apply to solutions of cohomological equations posed on Hénon-like mappings and a wide variety of non-uniformly hyperbolic systems. We note that Corollary 1 of [4, Theorem 1] is not correct — the solution is Hölder only on M_k and TM_k rather than T^jM_k for $j > 1$ as stated for reasons similar to those given above for the result in Nicol et al. [15].

2 Main results

We first describe one-dimensional Gibbs–Markov maps. Let $I \subset \mathbb{R}$ be a bounded interval, and \mathcal{P} a countable partition of I into intervals. We let m denote Lebesgue measure. Let $T: I \rightarrow I$ be a piecewise C^k , $k \geq 2$, expanding map such that T is C^k on the interior of each element of \mathcal{P} with $|T'| > \lambda > 1$, and for each $\alpha \in \mathcal{P}$, $T\alpha$ is a union of elements in \mathcal{P} . Let $P_n := \bigvee_{j=0}^n T^{-j}\mathcal{P}$ and $J_T := \frac{d(m \circ T)}{dm}$. We assume:

- (i) (Big images property) There exists $C_1 > 0$ such that $m(T\alpha) > C_1$ for all $\alpha \in \mathcal{P}$.
- (ii) There exists $0 < \gamma_1 < 1$ such that $m(\beta) < \gamma_1^n$ for all $\beta \in P_n$.
- (iii) (Bounded distortion) There exists $0 < \gamma_2 < 1$ and $C_2 > 0$ such that $|1 - \frac{J_T(x)}{J_T(y)}| < C_2\gamma_2^n$ for all $x, y \in \beta$ if $\beta \in P_n$.

Under these assumptions T has an invariant absolutely continuous probability measure μ and the density of μ , $h = \frac{d\mu}{dm}$ is bounded above and below by a constant $0 < C^{-1} \leq h(x) \leq C$ for m a.e. $x \in I$.

Note that a Markov map satisfies (i), (ii) and (iii) for finite partition \mathcal{P} .

It is proved in [15] for the Markov case (finite \mathcal{P}), and in [7] for the Gibbs–Markov case (countable \mathcal{P}) that if $\phi: I \rightarrow \mathbb{R}$ is Hölder continuous or Lipschitz continuous, and $\phi = \chi \circ T - \chi$ for some measurable function $\chi: I \rightarrow \mathbb{R}$, then there exists a function $\chi_0: I \rightarrow \mathbb{R}$ that is Hölder or Lipschitz on each of the elements of \mathcal{P} respectively, and $\chi_0 = \chi$ holds μ (or m) a.e. A related result to [7] is given in [4, Theorem 7] where T is the base map of a Young Tower, which has a Gibbs–Markov structure.

Fried [6] has shown that the transfer operator of a graph directed Markov system with $C^{k,\alpha}$ -contractions, acting on a space of $C^{k,\alpha}$ -functions, has a spectral gap. If we apply his result to our setting, letting the contractions be the inverse branches of a Gibbs–Markov map we can conclude that the transfer operator of a Gibbs–Markov map acting on C^k -functions has a spectral gap. As in Jenkinson’s paper [10] and with the same proof, this gives us immediately the following proposition, which is implied by the results of Fried and Jenkinson:

Proposition 1. Let $T: T \rightarrow I$ be a mixing Gibbs–Markov map such that T is C^k on each partition element and $T^{-1}: T(\alpha) \rightarrow \alpha$ is C^k on each partition element $\alpha \in \mathcal{P}$. Let $\phi: I \rightarrow \mathbb{R}$ be uniformly C^k on each of the partition elements $\alpha \in \mathcal{P}$. Suppose $\chi: I \rightarrow \mathbb{R}$ is a measurable function such that $\phi = \chi \circ T - \chi$. Then there exists a function $\chi_0: I \rightarrow \mathbb{R}$ such that χ_0 is uniformly C^k on $T\alpha$ for each partition element of $\alpha \in \mathcal{P}$, and $\chi_0 = \chi$ almost everywhere.

3 A counterexample

We remark that in general, if $\phi = \chi \circ T - \chi$, one cannot expect χ to be continuous on I if ϕ is C^k on I . We give an example of a Markov map T with Markov partition \mathcal{P} , a function ϕ that is C^k on I , and a function χ that is C^k on each element α of \mathcal{P} such that $\phi = \chi \circ T - \chi$, yet χ has no version that is continuous on I .

Let $0 < c < \frac{1}{4}$. Put $d = 2 - 4c$. Define $T: [0, 1] \rightarrow [0, 1]$ by

$$T(x) = \begin{cases} 2x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{4} \\ d(x - \frac{1}{2}) + \frac{1}{2} & \text{if } \frac{1}{4} < x < \frac{3}{4} \\ 2x - \frac{3}{2} & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}.$$

Figure 1: The graph of T .

Figure 3: The graph of $\chi \circ T$.

Figure 2: The graph of χ .

Figure 4: The graph of $\phi = \chi \circ T - \chi$.

If $c = \frac{1}{8}$, then the partition

$$\mathcal{P} = \left\{ \left[0, \frac{1}{8}\right], \left[\frac{1}{8}, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2} - \frac{1}{4d}\right], \left[\frac{1}{2} - \frac{1}{4d}, \frac{1}{2}\right], \right. \\ \left. \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{4d}\right], \left[\frac{1}{2} + \frac{1}{4d}, \frac{3}{4}\right], \left[\frac{3}{4}, \frac{7}{8}\right], \left[\frac{7}{8}, 1\right] \right\}$$

is a Markov partition for T . Define χ such that χ is 0 on $[\frac{1}{2} - \frac{1}{4d}, \frac{1}{2}]$ and 1 on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{4d}]$. On $[0, \frac{1}{4})$ we define χ so that $\chi(0) = 1$ and $\lim_{x \rightarrow \frac{1}{4}} \chi(x) = 0$, and on $(\frac{3}{4}, 1]$ we define χ so that $\chi(1) = 0$ and $\lim_{x \rightarrow \frac{3}{4}} \chi(x) = 1$. For any natural number k , this can be done so that χ is C^k except at the point $\frac{1}{2}$ where it has a jump. One easily check that ϕ defined by $\phi = \chi \circ T - \chi$ is C^k . This is illustrated in Figures 1–4.

4 Livšic theorems for piecewise expanding maps of an interval

Let $I = [0, 1)$ and let m denote Lebesgue measure on I . We consider piecewise expanding maps $T: I \rightarrow I$, satisfying the following assumptions:

- (i) There is a number $\lambda > 1$, and a finite partition \mathcal{P} of I into intervals, such that the restriction of T to any interval in \mathcal{P} can be extended to a C^2 -function on the closure, and $|T'| > \lambda$ on this interval.
- (ii) T has an absolutely continuous invariant measure μ with respect to which T is mixing.
- (iii) T has the property of being weakly covering, as defined by Liverani in [12], namely that there exists an n_0 such that for any element $\alpha \in \mathcal{P}$

$$\bigcup_{j=0}^{n_0} T^j(\alpha) = I.$$

For any $n \geq 0$ we define the partition $\mathcal{P}_n = \mathcal{P} \vee \dots \vee T^{-n+1}\mathcal{P}$. The partition elements of \mathcal{P}_n are called n -cylinders, and \mathcal{P}_n is called the partition of I into n -cylinders.

We prove the following two theorems.

Theorem 1. *Let (T, I, μ) be a piecewise expanding map satisfying assumptions (i), (ii) and (iii). Let $\phi: I \rightarrow \mathbb{R}$ be a Hölder continuous function, such that $\phi = \chi \circ T - \chi$ for some measurable function χ , with $e^{-\chi} \in L_1(m)$. Then there exists a function χ_0 such that χ_0 has bounded variation and $\chi_0 = \chi$ almost everywhere.*

For the next theorem we need some more definitions. Let A be a set, and denote by $\text{int } A$ the interior of the set A . We assume that the open sets $T(\text{int } \alpha)$, where α is an element in \mathcal{P} , cover $\text{int } I$.

We will now define a new partition \mathcal{Q} . For a point x in the interior of some element of \mathcal{P} , we let $Q(x)$ be the largest open set such that for any $x_2 \in Q(x)$, and any m -cylinder C_m , there are points $(y_{1,k})_{k=1}^n$ and $(y_{2,k})_{k=1}^n$, such that $y_{1,k}$ and $y_{2,k}$ are in the same element of \mathcal{P} , $T(y_{i,k+1}) = y_{i,k}$, $T(y_{1,1}) = x$, $T(y_{2,1}) = x_2$, and $y_{1,n}, y_{2,n} \in C_m$. (This forces $n \geq m$.)

Note that if $Q(x) \cap Q(y) \neq \emptyset$, then for $z \in Q(x) \cap Q(y)$ we have $Q(z) = Q(x) \cup Q(y)$. We let \mathcal{Q} be the coarsest collection of connected sets, such that any element of \mathcal{Q} can be represented as a union of sets $Q(x)$.

Theorem 2. *Let (T, I, μ) be a piecewise expanding map satisfying assumptions (i), (ii) and (iii). If $\phi: I \rightarrow \mathbb{R}$ is a continuously differentiable function, such that $\phi = \chi \circ T - \chi$ for some function χ with $e^{-\chi} \in L_1$, then there exists a function χ_0 such that χ_0 is continuously differentiable on each element of \mathcal{Q} and $\chi_0 = \chi$ almost everywhere. If T' is constant on the elements of \mathcal{P} , then χ_0 is piecewise C^k on \mathcal{Q} if ϕ is in C^k . If for each r , $\frac{1}{(T^r)'} is in C^k with derivatives up to order k uniformly bounded, then χ_0 is piecewise C^k on \mathcal{Q} if ϕ is in C^k .$*

It is not always clear how big the elements in the partition \mathcal{Q} are. The following lemma gives a lower bound on the diameter of the elements in \mathcal{Q} .

Lemma 1. *Assume that the sets $\{T(\text{int } \alpha) : \alpha \in \mathcal{P}\}$ cover $(0, 1)$. Let δ be the Lebesgue number of the cover. Then the diameter of $Q(x)$ is at least $\delta/2$ for all x .*

Proof. Let C_m be a cylinder of generation m . We need to show that for some $n \geq m$ there are sequences $(y_{1,k})_{k=1}^n$ and $(y_{2,k})_{k=1}^n$ as in the definition of \mathcal{Q} above.

Take n_0 such that $\mu(T^{n_0}(C_m)) = 1$. Write C_m as a finite union of cylinders of generation n_0 , $C_m = \bigcup_i D_i$. Then $R := [0, 1] \setminus T^{n_0}(\bigcup_i \text{int } D_i)$ consists of finitely many points. Let ε be the smallest distance between two of these points.

Let I_δ be an open interval of diameter δ . Let n_1 be such that $\delta \lambda^{-n_1} < \varepsilon$. Consider the full pre-images of I_δ under T^{n_1} . By the definition of δ , there is

at least one such pre-image, and any such pre-image is of diameter less than ε . Hence any pre-image contains at most one point from R .

If the pre-image does not contain any point of R , then I_δ is contained in some element of \mathcal{Q} and we are done. Assume that there is a point z in I_δ corresponding to the point of R in the pre-image of I_δ . Assume that z is in the right half of I_δ . The case when z is in the left part is treated in a similar way. Take a new open interval J_δ of length δ , such that the left half of J_δ coincides with the right half of I_δ .

Arguing in the same way as for I_δ , we find that a pre-image of J_δ contains at most one point of R . If there is no such point, or the corresponding point $z_J \in J_\delta$ is not equal to z , then $I_\delta \cup J_\delta$ is contained in an element in \mathcal{Q} and we are done.

It remains to consider the case $z = z_J$. Let $I_\delta = (a, b)$ and $J_\delta = (c, d)$. Then the intervals (a, z) and (z, d) are both of length at least $\delta/2$, and both are contained in some element of \mathcal{Q} . This finishes the proof. \square

Corollary 1. *If $\beta > 1$ and $T: x \mapsto \beta x \pmod{1}$ is a β -transformation then clearly T is weakly covering and $\mathcal{Q} = \{(0, 1)\}$, so in this case Theorem 2 and Theorem 1 of [15] imply that χ_0 is in C^k if ϕ is in C^k .*

Remark 1. If $T: x \mapsto \beta x + \alpha \pmod{1}$ is an affine β -transformation, then $\mathcal{Q} = \{(0, 1)\}$, and hence if e^{-x} is in $L_1(m)$ then χ has a C^k version.

5 Proof of Theorem 1

We continue to assume that (T, I, μ) is a piecewise expanding map satisfying assumptions (i), (ii) and (iii). For a function $\psi: I \rightarrow \mathbb{R}$ we define the weighted transfer operator \mathcal{L}_ψ by

$$\mathcal{L}_\psi f(x) = \sum_{T(y)=x} e^{\psi(y)} \frac{1}{|d_y T|} f(y).$$

The proof is based on the following two facts, that can be found in Hofbauer and Keller's papers [8, 9]. The first fact is

There is a function $h \geq 0$ of bounded variation such that if $f \in L^1$ with $f \geq 0$ and $f \neq 0$, then $\mathcal{L}_0^n f$ converges to $h \int f \, dm$ in L^1 . (2)

The second fact is

Let $f \in L^1$ with $f \geq 0$ and $f \neq 0$ be fixed. There is a function $w \geq 0$ with bounded variation, a measure ν , and a number $a > 0$, depending on ϕ , such that

$$a^n \mathcal{L}_\phi^n f \rightarrow w \int f \, d\nu, \quad (3)$$

in L^1 .

For f of bounded variation, these facts are proved as follows. Theorem 1 of [8] gives us the desired spectral decomposition for the transfer operator acting on functions of bounded variation. Proposition 3.6 of Baladi's book [3] gives us that there is a unique maximal eigenvalue. This proves the two facts for f of bounded variation. The case of a general f in L^1 follows since such an f can be approximated by functions of bounded variation.

Using that T is weakly covering, we can conclude by Lemma 4.2 in [12], that $h > \gamma > 0$. The proof of this fact in [12] goes through also for w , and so we may also conclude that $w > \gamma > 0$.

Let us now see how Theorem 1 follows from these facts. The following argument is analogous to the argument used by Pollicott and Yuri in [23] for β -expansions. We first observe that $\phi = \chi \circ f - \chi$ implies that

$$\begin{aligned} \mathcal{L}_\phi^n 1(x) &= \sum_{T^n(y)=x} e^{S_n \phi(y)} \frac{1}{|d_y T^n|} = \sum_{T^n(y)=x} e^{\chi(T^n y) - \chi(y)} \frac{1}{|d_y T^n|} \\ &= e^{\chi(x)} \sum_{T^n(y)=x} e^{-\chi(y)} \frac{1}{|d_y T^n|} = e^{\chi(x)} \mathcal{L}_0^n e^{-\chi}(x). \end{aligned}$$

Since $a^n \mathcal{L}_\phi^n 1 \rightarrow w$ and $e^{-\chi} \mathcal{L}_\phi^n 1 = \mathcal{L}_0^n e^{-\chi} \rightarrow h \int e^{-\chi} \, dm$ we have that $a^n \mathcal{L}_\phi^n 1$ converges to w in L^1 and $\mathcal{L}_\phi^n 1$ converges to $h e^\chi \int e^{-\chi} \, dm$ in L^1 . By taking a subsequence, we can achieve that the convergences are a.e. Therefore, we must have $a = 1$ and

$$w(x) = e^{\chi(x)} h(x) \int e^{-\chi} \, dm, \quad \text{a.e.}$$

It follows that

$$\chi(x) = \log w(x) - \log \int e^{-\chi} \, dm - \log h(x),$$

almost everywhere. Since h and w are bounded away from zero, their logarithms are of bounded variation. This proves the theorem.

6 Proof of Theorem 2

We first note that it is sufficient to prove that χ_0 is continuously differentiable on elements of the form $Q(x)$.

Let x and y satisfy $T(y) = x$. Then by $\phi = \chi \circ T - \chi$ we have $\chi(x) = \phi(y) + \chi(y)$.

Let x_1 be a point in an element of \mathcal{Q} , and take $x_2 \in Q(x_1)$. We choose pre-images $y_{1,j}$ and $y_{2,j}$ of x_1 and x_2 such that $T(y_{1,j}) = x_1$ and $T(y_{2,j}) = y_{1,j-1}$. We then have

$$\chi(x_1) - \chi(x_2) = \sum_{j=1}^n (\phi(y_{1,j}) - \phi(y_{2,j})) + \chi(y_{1,n}) - \chi(y_{2,n}).$$

We would like to let $n \rightarrow \infty$ and conclude that $\chi(y_{1,n}) - \chi(y_{2,n}) \rightarrow 0$. By Theorem 1 we know that χ has bounded variation. Assume for contradiction that no matter how we choose $y_{1,j}$ and $y_{2,j}$ we cannot make $|\chi(y_{1,n}) - \chi(y_{2,n})|$ smaller than some $\varepsilon > 0$. Let m be large and consider the cylinders of generation m . For any such cylinder C_m , we can choose $y_{1,j}$ and $y_{2,j}$ such that $y_{1,n}$ and $y_{2,n}$ both are in C_m . Since $|\chi(y_{1,n}) - \chi(y_{2,n})| \geq \varepsilon$, the variation of χ on C_m is at least ε . Summing over all cylinders of generation m , we conclude that the variation of χ on I is at least $N(m)\varepsilon$. Since m is arbitrary and $N(m) \rightarrow \infty$ as $m \rightarrow \infty$, we get a contradiction to the fact that χ is of bounded variation.

Hence we can make $|\chi(y_{1,n}) - \chi(y_{2,n})|$ smaller than any $\varepsilon > 0$ by choosing $y_{1,j}$ and $y_{2,j}$ in an appropriate way. We conclude that

$$\chi(x_1) - \chi(x_2) = \sum_{j=1}^{\infty} (\phi(y_{1,j}) - \phi(y_{2,j})).$$

If $x_1 \neq x_2$ then $y_{1,j} \neq y_{2,j}$ for all j , and we have

$$\frac{\chi(x_1) - \chi(x_2)}{x_1 - x_2} = \sum_{j=1}^{\infty} \frac{\phi(y_{1,j}) - \phi(y_{2,j})}{y_{1,j} - y_{2,j}} \frac{y_{1,j} - y_{2,j}}{x_1 - x_2}.$$

Clearly, the limit of the right hand side exists as $x_2 \rightarrow x_1$, and is

$$\sum_{j=1}^{\infty} \phi'(y_{1,j}) \frac{1}{(T^j)'(y_{1,j})}.$$

The series converges since $|(T^j)'| > \lambda^j$. This shows that $\chi'(x_1)$ exists and satisfies

$$\chi'(x_1) = \sum_{j=1}^{\infty} \phi'(y_{1,j}) \frac{1}{(T^j)'(y_{1,j})}. \quad (4)$$

If T' is constant on the elements of \mathcal{P} , then (4) implies that χ is in C^k provided that ϕ is in C^k .

Let us now assume that $\frac{1}{(T^r)'} is in C^k with derivatives up to order k uniformly bounded in r . We proceed by induction. Let $g_n = \frac{1}{(T^n)'}$. Assume that$

$$\chi^{(m)}(x) = \sum_{n=1}^{\infty} \psi_{n,m}(y_n) g_n(y_n), \quad (5)$$

where $(\psi_{n,m})_{n=1}^{\infty}$ is in C^{n-m} with derivatives up to order $n - m$ uniformly bounded. Then

$$\chi^{(m+1)}(x) = \sum_{n=1}^{\infty} (\psi'_{n,m}(y_n) g_n(y_n) + \psi_{n,m}(y_n) g'_n(y_n)) g_n(y_n) = \sum_{n=1}^{\infty} \psi_{n,m+1} g_n(y_n).$$

This proves that there are uniformly bounded functions $\psi_{n,m}$ such that (5) holds for $1 \leq m \leq k$. The series in (5) converges uniformly since g_n decays with exponential speed. This proves that χ is in C^k . \square

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