On symmetric attractors in reversible dynamical systems

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Abstract

Let $\Gamma \subset O(n)$ be a finite group acting orthogonally on $\mathbb{R}^n$. We say that Γ is a reversing symmetry group of a homeomorphism, diffeomorphism or flow $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$ ($t \in \mathbb{Z}$ or $t \in \mathbb{R}$) if Γ has an index two subgroup $\tilde{\Gamma}$ whose elements commute with f^t and for all elements $\rho \in \Gamma - \tilde{\Gamma}$ and all t , $f^t \circ \rho(x) = \rho \circ f^{-t}(x)$. We give necessary group and representation theoretic conditions for subgroups of reversing symmetry groups to occur as symmetry groups of attractors (Liapunov stable ω-limit sets). These conditions arise due to topological obstructions.

In dimensions 1 and 2 we present a complete description of possible symmetry groups of asymptotically stable attractors for homeomorphisms and diffeomorphisms (these attractors cannot possess reversing symmetries). We also have a fairly complete description in the context of subgroups which contain reversing symmetries. For all dimensions n we present complete results on the possible symmetry groups of connected asymptotically stable attractors.

In addition, we summarize our results in the context of reversible equivariant flows, which are complete in dimensions 1 and 2 as well as for subgroups of $\tilde{\Gamma} \subset O(n)$ in any dimension n. A survey of the corresponding results in the equivariant context is also given.

Contents

1 Introduction and outline.

Several studies [2, 6, 18, 19] have led to necessary conditions for ω-limit sets to have symmetries in equivariant dynamical systems (necessary and sufficient if the ω-limit sets are Liapunov stable). Reversing symmetries occur in many applications of current interest, cf. e.g. [23, 13]. In this paper we discuss extensions of the equivariant results by taking into account reversing symmetries.

We consider continuous invertible dynamical systems of the form $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$, with the time variable $t \in \mathbb{Z}$ (discrete time) or $t \in \mathbb{R}$ (continuous time), such that f^t is continuous on $\mathbb{R}^n$, $f^{t_1} \circ f^{t_2} = f^{t_1+t_2}$ for all t_1, t_2, and $f^0 = \mathrm{id}$. In the case of discrete time, the dynamics is generated by a single map that we normally note by f (omitting the superscript). We distinguish between homeomorphisms (invertible continuous maps) and diffeomorphisms (differentiable homeomorphisms), although all the results we obtain in this paper apply to both categories.

The ω-limit set $\omega(\mathbf{x})$ of a point $\mathbf{x} \in \mathbb{R}^n$ consists of the closure of the set of accumulation points of $\{f^t(x)\}$.

In the context of this paper, we define an *attractor* to be a Liapunov stable ω-limit set. A set A is called *Liapunov stable* if for any open neighbourhood U of A there exists an open neighbourhood V of A such that $f^j(V) \subset U$ for all $j \geq 0$. An attractor A is called *asymptotically stable* if there exists an open neighbourhood U of A such that $\omega(x) \subseteq A$ for all $\mathbf{x} \in U$.

As in [15] we discuss two types of symmetry properties. The homeomorphism (continuous invertible map) $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a *symmetry* of f^t if for all t

$$f^t \circ \sigma = \sigma \circ f^t, \tag{1}$$

where $\circ$ denotes composition. When f^t has a symmetry σ it is also called *σ-equivariant.* The collection of all symmetries of f^t forms a group under composition. This group is usually called the *symmetry group* of f^t.

A homeomorphism $\rho : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a *reversing symmetry* of f^t if for all t

$$f^t \circ \rho = \rho \circ f^{-t}. \tag{2}$$

As the composition of two reversing symmetries yields a symmetry and the composition of a reversing symmetry and a symmetry yields a reversing symmetry, the union of the sets of symmetries and reversing symmetries of f^t form a group, which we will call the *reversing symmetry group* of f^t. However, note that the statement that f has symmetry group $\Gamma \subset O(n)$ does not mean that f does not commute with any homeomorphism not contained in Γ, merely that we are restricting our discussion to elements of Γ (and similarly when we say that f has reversing symmetry group Γ).

It is easily checked that any reversing symmetry group containing at least one reversing symmetry contains a normal subgroup of index two which contains only symmetries. A dynamical system which possesses a reversing symmetry will be called *reversible.* Note that we do not require reversing symmetries ρ to be involutions in the sense that $\rho^2 = \mathrm{id}$.

The main focus in this paper will be on attractors (i.e. Liapunov stable ω-limit sets) of dynamical systems in $\mathbb{R}^n$ having a reversing symmetry group Γ that is a finite subgroup of $O(n)$ acting as a linear, orthogonal representation on $\mathbb{R}^n$.

The *symmetry group* $\Sigma \leq \Gamma$ of a subset $A \subset \mathbb{R}^n$ is defined as

$$\Sigma = \{\gamma \in \Gamma \mid \gamma(A) = A\}. \tag{3}$$

Let $T \leq \Sigma$ denote the subgroup of instantaneous symmetries, i.e. the symmetries which fix A pointwise. In describing the symmetry of the set A it is relevant to know both Σ and T. We indicate the symmetry of A by the pair (Σ, T).

Definition 1.1 (Admissible subgroups) *Suppose that Σ is a subgroup of Γ. We say that Σ is admissible for homeomorphisms (diffeomorphisms, flows) if there exists a homeomorphism (diffeomorphism, flow) with reversing symmetry group Γ and an attractor with symmetry $(\Sigma, 1)$.*

In this paper the central question we address is: [1]

> *What subgroups of reversing symmetry groups are admissible as symmetry groups of attractors (Liapunov stable ω-limit sets) for homeomorphisms, diffeomorphisms and flows ?*

Later, in Section 2.4 we will show how our results extend to attractors inside fixed point subspaces, i.e. with symmetry (Σ, T) and $T \neq 1$.

Before stating the results of this paper in Section 2, we give some background on the results obtained in the equivariant context.

1.1 Review of results in the equivariant context.

Suppose that f^t commutes with a finite group $\Gamma \subset O(n)$. One of the key early results was that the symmetry group of a periodic orbit must be a cyclic extension of an isotropy subgroup (see [19]). For general ω-limit sets the description of admissible symmetries was not so simple yet still a fairly complete picture emerged, particularly for Liapunov stable ω-limit sets. In the continuous category necessary and sufficient conditions were given for admissibility of Liapunov stable ω-limit sets by the work of Melbourne *et al.* [19] and Ashwin and Melbourne [2]. It was found that group elements which act as reflections play the crucial role in determining admissibility. In this context we call $\tau : \mathbb{R}^n \mapsto \mathbb{R}^n$ a reflection if $\tau^2 = \mathrm{id}$ and $\dim \mathrm{Fix}(\tau) = n - 1$. The fixed point subspace of τ is called a *reflection hyperplane* and divides $\mathbb{R}^n$ into two disjoint connected components. In order to state the main results of [19] and [2] we introduce the following notation. If Δ is a subgroup of Γ we let L_Δ denote the union of reflection hyperplanes corresponding to group elements of Γ not contained in Δ. The main result of [19] was that if Σ is admissible then there is a subgroup Δ such that

a) Δ is a normal subgroup of Σ,

b) Σ/Δ is cyclic,

c) Δ fixes a connected component of $\mathbb{R}^n - L_\Delta$.

In [2], Ashwin and Melbourne showed that this representation-theoretic condition is also sufficient by using a construction involving Cayley graphs. In particular they showed that all cyclic subgroups of Γ were admissible, recapturing a result of King and Stewart [11]. They also showed that if Σ is to be connected then Σ must fix a connected component of $\mathbb{R}^n - L_\Sigma$. Furthermore the attractors constructed in the proof of this theorem were

[1] The setting of Liapunov stable ω-limit sets is chosen in connection with the structure theorems of previous work on symmetries of attractors [2, 6, 18, 19]. However, in the reversible case, other types of ω-limit sets may be equally (or more) relevant, see also the discussion in Section 3.

(a)

Γ	Σ
$\mathbb{Z}_2$	1

(b)

Γ	Σ
$\mathbb{Z}_m$	$\mathbb{Z}_m$
$\mathbf{D}_m$	1

Table 1: Subgroups Σ of symmetry group Γ which are admissible for flows in $\mathbb{R}^n$ in the case that Γ is a finite subgroup of $O(n)$ for (a) $n = 1$ and (b) $n = 2$ (taken from [6]). $\mathbb{Z}_m$ and $\mathbf{D}_m$ denote the natural representations of these groups in $\mathbb{R}^2$ and $\mathbb{R}$. In $\mathbb{R}$, $\mathbb{Z}_2$ denotes the group generated by -1. In $\mathbb{R}^2, \mathbb{Z}_m$ denotes the cyclic group generated by an m-fold rotation around the origin, whereas $\mathbf{D}_m$ denotes the representation of the dihedral group generated by an m-fold rotation around the origin and a reflection in a straight line through the origin.

topologically transitive, possessed open basins of attraction and the connected attractors were topologically mixing [2].

When equivariant homeomorphisms or flows are considered then there are greater restrictions on admissibility. Let L denote the union of the reflection hyperplanes corresponding to reflections in Γ. Melbourne [18] showed that if A is a Σ-symmetric ω-limit set for a Γ-equivariant homeomorphism then either

- Σ contains no reflections and fixes a connected component of $\mathbb{R}^n - L$, or
- Σ has an index two subgroup that fixes a connected component of $\mathbb{R}^n - L$.

If Σ is to be admissible for flows then Σ must fix a connected component of $\mathbb{R}^n - L$. This severe restriction is forced by the easy observation that if τ is a reflection and f is a homeomorphism then either f fixes both connected components of $\mathbb{R}^n - L$ or it interchanges both components. In the flow case both components must be fixed.

Precise necessary and sufficient conditions for admissibility of flows and homeomorphisms are given in [6]. In fact a subgroup Σ is admissible for homeomorphisms if and only if Σ is of Class I or Class II. Implicitly, a definition of these classes is given in Definition 2.3 and Definition 2.4. The ω-limit sets that realise admissibility can be taken to be asymptotically stable and if $n \geq 4$ they may be taken to be Axiom A.

The sufficiency conditions for flows is complicated somewhat by a topological obstruction in the case $n = 2$, but otherwise if Σ is of Class I then it is admissible and the admissibility can be realised by asymptotically stable ω-limit sets (which can be taken as Axiom A if $n \geq 5$). In the case of dimension 2 the only subgroup of $\mathbf{D}_m$ that is admissible for flows is 1 and if $\Gamma = \mathbb{Z}_m$ then the only admissible subgroups are 1 and $\mathbb{Z}_m$.

The equivariant results in dimension 1 and 2 are summarised in Tables 1 and 2. For the equivariant results in $\mathbb{R}^3$, see [6].

2 Symmetric attractors in reversible systems.

We first introduce some notation and terminology. We consider homeomorphisms and flows in $\mathbb{R}^n$ with reversing symmetry group Γ. We let $\tilde{\Gamma}$ denote the symmetry group associated to Γ, i.e. $\tilde{\Gamma}$ is the largest subgroup of Γ containing no reversing symmetries. Then, $\tilde{\Gamma}$ is a normal subgroup of Γ and when $\Gamma \neq \tilde{\Gamma}$ it is a subgroup of index two.

(a)

Γ	Σ	
$\mathbb{Z}_2$	1	Class I
	$\mathbb{Z}_2$	Class II

(b)

Γ	Σ	
$\mathbb{Z}_m$	$\mathbb{Z}_k$	Class I
$\mathbf{D}_m,\ m$ odd	1	Class I
	$\mathbf{D}_1$	Class II
$\mathbf{D}_2$	1	Class I
	$\mathbf{D}_1,\ \mathbb{Z}_2$	Class II
$\mathbf{D}_m,\ m \geq 4$ even	1	Class I
	$\mathbb{Z}_2$	Class II

Table 2: Subgroups Σ of symmetry group Γ which are admissible for homeomorphisms and diffeomorphisms in $\mathbb{R}^n$ in the case that Γ is a finite subgroup of $O(n)$ with (a) $n = 1$ and (b) $n = 2$ (taken from [6]). Notation as in Table 1. m is a positive integer and k any positive integer that divides m.

Liapunov stability confers important restrictions on the symmetry properties of ω-limit sets. First of all, in [15] we obtain a reversible extension of a result of [4, 19].

Theorem 2.1 ([15]) *Suppose $f^t : \mathbb{R}^n \to \mathbb{R}^n$ has a Liapunov stable ω-limit set A. Suppose γ is either a symmetry or a reversing symmetry of f^t. Then if $\gamma(A) \cap A \neq \emptyset$ then $\gamma(A) = A$.*

Moreover, Liapunov stability also imposes restrictions on the topological dynamics of ω-limit sets that are symmetric with respect to a reversing symmetry. Recall that a set A is called transitive if $A = \omega(\mathbf{x})$ for some $\mathbf{x} \in A$.

Theorem 2.2 ([15]) *Suppose that A is a Liapunov stable ω-limit set of a dynamical system f^t with reversing symmetry ρ. Then,*

$$\rho(A) = A \quad \Rightarrow \quad A \textit{ is transitive and not asymptotically stable.}$$

In contrast to Theorem 2.1, Theorem 2.2 has no equivariant analogue.

2.1 Restrictions

In equivariant dynamical systems restrictions on symmetries for ω-limit sets occur due to the presence of symmetries which are reflections. In analogy to the equivariant case we let $\tilde{L}$ denote the union of all the reflection hyperplanes of $\tilde{\Gamma}$, i.e.

$$\tilde{L} := \bigcup_{\text{refl } \tau \in \tilde{\Gamma}} \mathrm{Fix}(\tau). \tag{4}$$

It should be noted that fixed point subspaces of reflections in $\Gamma - \tilde{\Gamma}$ (i.e. that are reversing symmetries) are explicitly not included in $\tilde{L}$. We also define,

$$L := \bigcup_{\text{refl } \tau \in \Gamma} \mathrm{Fix}(\tau). \tag{5}$$

As a last preliminary, we recall that the *centralizer* of Γ in $O(n)$ is defined as $C(\Gamma) := \{\sigma \in O(n) \mid \sigma \circ \gamma = \gamma \circ \sigma,\ \forall \gamma \in \Gamma\}$.

In analogy to the ideas leading to restrictions on symmetries of ω-limit sets in the equivariant case, in the reversible context we find necessary conditions involving the following notion of Class I and Class II subgroups. Recall that if Δ is a subgroup of Γ then L_Δ denotes the union of reflection hyperplanes corresponding to group elements not contained in Δ, and $\tilde{\Delta} := \Delta \cap \tilde{\Gamma}$.

Definition 2.3 (Class I subgroups) *A subgroup $\Sigma \leq \Gamma$ is of Class I if*

(a) Σ fixes a connected component $\tilde{C}$ of $I\!R^n - \tilde{L}$

(b) Σ has a subgroup Δ such that

(i) Δ fixes a connected component C of $I\!R^n - L_\Delta$ (contained in the connected component $\tilde{C}$ of $I\!R^n - \tilde{L}$.)

(ii) $\tilde{\Delta}$ is a normal subgroup of $\tilde{\Sigma}$ and Σ.

(iii) $\tilde{\Sigma}/\tilde{\Delta} \simeq \mathbb{Z}_k$, and
$\Sigma/\tilde{\Delta} \simeq \mathbf{D}_k$ if $\Sigma \neq \tilde{\Sigma}$

Definition 2.4 (Class II subgroups) *A subgroup $\Sigma \leq \Gamma$ is of Class II if*

(a) Σ is not of Class I

(b) Σ has a normal subgroup Σ' that is of Class I and

(i) $\Sigma/\Sigma' \simeq \mathbb{Z}_2$.

(ii) there exists an involution $B \in C(\Gamma)$ such that $\sigma(\tilde{C}) = B(\tilde{C})$ for all $\sigma \in \Sigma - \Sigma'$, with $\tilde{C}$ the connected component corresponding to the Class I property of Σ'.

(iii) Let Δ be the subgroup of Σ' corresponding to the Class I property of Σ'. Then,
$\tilde{\Sigma}/\tilde{\Delta} \simeq \mathbb{Z}_{2k}$, and
$\Sigma/\tilde{\Delta} \simeq \mathbf{D}_{2k}$ if $\Sigma \neq \tilde{\Sigma}$

Remark 2.5 *If Γ does not contain reversing symmetries, then we recapture from Definition 2.3 and Definition 2.4 the definition of Class I and Class II subgroups as given in [6].*

Our first theorem establishes necessary conditions for the occurrence of subgroups of Γ as symmetry groups of Liapunov Stable ω-limit sets.

Theorem 2.6 (Necessary conditions for admissibility) *Suppose Γ is a finite subgroup of $O(n)$ ($n \geq 1$) and $\Sigma \leq \Gamma$ is an admissible subgroup for homeomorphisms $f : I\!R^n \mapsto I\!R^n$ with reversing symmetry group Γ. Then Σ must be of Class I or of Class II. In the case of a flow in $I\!R^n$, Σ must be of Class I.*

In case Γ does not contain reversing symmetries it has been shown in [6] that subgroups Σ are admissible for homeomorphisms and diffeomorphisms if and only if they are of Class I or of Class II and that in the case of flows all Class I subgroups are admissible, whenever $n \geq 3$.

A full discussion of the possible symmetry groups of attractors for reversible equivariant systems is beyond the scope of the present paper.

In this paper, we will focus on the problem of admissibility in dimensions 1 and 2. We furthermore present results on admissible subgroups of connected attractors, and discuss the admissibility of subgroups for attractors inside (invariant) fixed point subspaces of symmetries.

2.2 Admissibility in dimensions 1 and 2

In dimensions 1 and 2 the situation is special, due to the extra topological restrictions that arise. On the other hand, the situation in low dimension is suitable for case-by-case analysis as the number of cases to be considered is not too large.

Theorem 2.7 (Admissibility in dimensions 1 and 2) *Let Γ be a finite subgroup of $O(n)$. If a subgroup $\Sigma(\tilde{\Sigma})$ is admissible for homeomorphisms and diffeomorphisms $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ with reversing symmetry group $\Gamma = \tilde{\Gamma} \cup \rho\tilde{\Gamma}$ and $n \in \{1, 2\}$, then it is listed in Table 3. For the subgroups in this table that are labelled* **undecided** *we have as yet no proof of (in)admissibility. All other listed subgroups are admissible, and the ω-limit sets that realize admissibility for these subgroups may be taken to be Liapunov stable periodic orbits.*

Remark 2.8 (Open problem) *The results in dimensions 1 and 2 as listed in Table 3 are complete in all cases except when $\Gamma(\tilde{\Gamma}) = \mathbb{Z}_{2m}(\mathbb{Z}_m)$ acting in $\mathbb{R}^2$, with $m > 1$. In that case the admissibility of $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_{2m}(\mathbb{Z}_m)$ is an open problem. If there exists an attractor with this symmetry property, then it must be connected but cannot be a (quasi-)periodic orbit.*

Remark 2.9 (Complete results for asymptotically stable attractors) *Because the problem discussed in Remark 2.8 is the only open one and involves a subgroup containing reversing symmetries, our results in dimensions 1 and 2 as listed in Table 3 are complete as far they concern subgroups that do not contain reversing symmetries. This means that Table 3 contains complete results on the admissibility of symmetry groups of asymptotically stable attractors in systems with a reversing symmetry group in dimensions 1 and 2.*

Remark 2.10 (Degenerate constructions) *The admissibility for some subgroups in Table 3 (those that are labelled with a *), can only be realized by a single involutory map $f = -\mathrm{id}$. Involutory maps are special, in the sense that every symmetry of such a map is also a reversing symmetry and vice-versa.*

For completeness we also mention the corresponding results on admissibility for flows.

Theorem 2.11 (Admissibility for flows in dimensions 1 and 2 [15]) *The admissible subgroups for flows in $\mathbb{R}^n$ with reversing symmetry groups that are finite subgroups of $O(n)$ with $n \in \{1, 2\}$ are listed in Table 4. The ω-limit sets that realize admissibility may be taken to be Liapunov stable periodic orbits.*

It should be noted that reversing symmetries have to be taken into account in order to determine the admissibility of symmetry groups that do not contain reversing symmetries themselves.

For example, consider a reversing symmetry group Γ with associated symmetry group $\tilde{\Gamma}$. Then, if a subgroup $\Sigma \leq \tilde{\Gamma}$ is admissible for $\tilde{\Gamma}$-equivariant homeomorphisms, this does not imply that Σ is admissible for a system with reversing symmetry group Γ. Namely, Σ may be a Class I or Class II subgroup of $\tilde{\Gamma}$ and at the same time not a Class I or Class II subgroup of Γ.

Example 2.12 *Suppose $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a $\mathbf{D}_2$-equivariant homeomorphism with $\mathbf{D}_2$ being generated by linear reflections in the coordinate axes then $\mathbf{D}_1$ (reflection in either of the two coordinate axes) is a Class II subgroup of $\mathbf{D}_2$ and as such admissible. If f is not only*

(a)

Γ ($\tilde{\Gamma}$)	Σ ($\tilde{\Sigma}$)	
$\mathbb{Z}_2$ (1)	$\mathbb{Z}_2$ $(1)^*$, 1(1)	Class I

(b)

Γ ($\tilde{\Gamma}$)	Σ ($\tilde{\Sigma}$)	
$\mathbb{Z}_2$ (1)	$\mathbb{Z}_2$ (1), 1 (1)	Class I
$\mathbb{Z}_{2m}$ ($\mathbb{Z}_m$), $m \geq 3$ odd	$\mathbb{Z}_2$ $(1)^*$, 1 (1)	Class I
	$\mathbb{Z}_{2m}$ ($\mathbb{Z}_m$) **undecided**	Class I
$\mathbb{Z}_{2m}$ ($\mathbb{Z}_m$), $m \geq 2$ even	$\mathbb{Z}_2$ ($\mathbb{Z}_2$), 1 (1)	Class I
	$\mathbb{Z}_{2m}$ ($\mathbb{Z}_m$) **undecided**	Class I
$\mathbf{D}_2$ ($\mathbf{D}_1$)	$\mathbf{D}_1$ (1), 1 (1)	Class I
	$\mathbf{D}_2$ ($\mathbf{D}_1$), $\mathbf{D}_1$ ($\mathbf{D}_1$), $\mathbb{Z}_2$ (1)	Class II
$\mathbf{D}_m$ ($\mathbb{Z}_m$)	$\mathbf{D}_k$ ($\mathbb{Z}_k$), $\mathbb{Z}_k$ ($\mathbb{Z}_k$)	Class I
$\mathbf{D}_{2m}$ ($\mathbf{D}_m$), $m \geq 3$ odd	$\mathbf{D}_1$ (1), 1 (1)	Class I
	$\mathbb{Z}_2$ $(1)^*$	Class II
$\mathbf{D}_{2m}$ ($\mathbf{D}_m$), $m \geq 2$ even	$\mathbf{D}_1$ (1), 1 (1)	Class I
	$\mathbb{Z}_2$ ($\mathbb{Z}_2$)	Class II

Table 3: All subgroups Σ of reversing symmetry group Γ (with $\Gamma \neq \tilde{\Gamma}$) that are (possibly) admissible for homeomorphisms and diffeomorphisms of $\mathbb{R}^n$ in the case that Γ is a finite subgroup of $O(n)$, with (a) $n = 1$ and (b) $n = 2$. The associated symmetry groups $\tilde{\Gamma}$ and $\tilde{\Sigma}$ of Γ and Σ are indicated between brackets. Other notation as in Table 2. For the two subgroups labelled **undecided**, admissibility is as yet undecided. For all other listed subgroups admissibility can be realized by periodic orbits. For the subgroups labelled with a *, admissibility can only be realized by the map $f = -\mathrm{id}$.

(a)

Γ ($\tilde{\Gamma}$)	Σ ($\tilde{\Sigma}$)
$\mathbb{Z}_2$ (1)	1 (1)

(b)

Γ ($\tilde{\Gamma}$)	Σ ($\tilde{\Sigma}$)
$\mathbb{Z}_{2m}$ ($\mathbb{Z}_m$), $n \geq 1$	1 (1)
$\mathbf{D}_1$ (1)	$\mathbf{D}_1$ (1)
$\mathbf{D}_m$ ($\mathbb{Z}_m$), $m \geq 1$	$\mathbf{D}_m$ ($\mathbb{Z}_m$), $\mathbf{D}_1$ (1), 1 (1)
$\mathbf{D}_{2m}$ ($\mathbf{D}_m$), $n \geq 1$	$\mathbf{D}_1$ ($\mathbf{D}_1$), 1 (1)

Table 4: Subgroups Σ of reversing symmetry groups Γ (with $\Gamma \neq \tilde{\Gamma}$) that are admissible for flows in $\mathbb{R}^n$ in the case that Γ is a finite subgroup of $O(n)$, with (a) $n = 1$ and (b) $n = 2$. Notation as in Table 3. (Taken from [15].)

$\mathbf{D}_2$*-equivariant, but possesses an additional reversing symmetry that is a linear reflection in a diagonal then the reversing symmetry group* $\Gamma = \mathbf{D}_4$ *with* $\tilde{\Gamma} = \mathbf{D}_2$*. In the latter case the subgroup* $\Sigma = \tilde{\Sigma} = \mathbf{D}_1$ *(one of the two groups generated by a reflection in a coordinate axis) is not a Class II subgroup of* Γ *as condition (b)(ii) of Definition 2.4 does not hold, and hence is not admissible.*

2.3 Connected attractors

We will discuss admissible Class I subgroups for which admissibility may be realized by a *connected attractor.* We follow Ashwin and Melbourne [2] and call such subgroups *strongly admissible.*

Theorem 2.13 (Strongly admissible subgroups) *Suppose* Γ *is a finite subgroup of* $O(n)$, $n \geq 3$ *and* $\Sigma \leq \Gamma$*. Then, a subgroup* Σ *that does not contain a reversing symmetry is strongly admissible for homeomorphisms and diffeomorphisms* $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ *with reversing symmetry group* Γ *if and only if* Σ *fixes a connected component of* $\mathbb{R}^n - L$*. The* ω*-limit sets that realize admissibility can be taken to be asymptotically stable. Moreover, if* $n \geq 4$ *the* ω*-limit sets that realize admissibility can be taken to be Axiom A attractors.*

Remark 2.14 *It should be noted that in the case of flows, attractors are always connected. In fact, the admissibility result for flows is just a special case of Theorem 2.13 (where Axiom A can be obtained in dimension* $n \geq 5$*)* [15].

It should be noted that in contrast to the situation in equivariant systems, in the reversible context not all admissible Class I subgroups are strongly admissible. Namely, our definition of Class I subgroups allows for subgroups that require the attractor to have disjoint connected components inside more than one connected component of $\mathbb{R}^n - L$. Some examples of this can be found in our discussion of the situation in dimensions 2.

2.4 Symmetric ω-limit sets inside fixed point subspaces

So far we have concentrated mainly on symmetric ω-limit sets with trivial instantaneous symmetry, i.e. we have studied the admissibility of subgroup (Σ, T) with $T = 1$. We now discuss how the results obtained in the previous sections extend to the case of symmetry groups (Σ, T) with $T \neq 1$, giving a reversible extension to the exposition in [6].

We first note T must be a normal subgroup of Σ and that T must be an isotropy subgroup [18].[2] Moreover, if T contains a reversing symmetry then the ω-limit set must be a fixed point or a periodic orbit of period two. The situation may become more interesting when T does not contain a reversing symmetry, and from now on we will assume that this is the case. Note that then $\mathrm{Fix}(T)$ is an invariant subspace for f^t.

Necessary conditions for admissibility of a pair of subgroups (Σ, T) are obtained by restricting to $\mathrm{Fix}(T)$, which is isomorphic to $\mathbb{R}^{n_T}$ for some $n_T \leq n$.

Let $N(T)$ denote the normalizer of T in Γ, i.e.

$$N(T) := \{\gamma \in \Gamma \mid \gamma \circ \tau \circ \gamma^{-1} \in T \text{ for all } \tau \in T\}. \tag{6}$$

[2] A group $T \leq \Gamma$ is called an isotropy subgroup of Γ if there exists a point $\mathbf{x} \in \mathbb{R}^n$ such that $T = \{\gamma \in \Gamma \mid \gamma(\mathbf{x}) = \mathbf{x}\}$.

A flow or diffeomorphism f^t on $\mathbb{R}^n$ with reversing symmetry group Γ restricts to a flow or diffeomorphism with reversing symmetry group $\Gamma_T := N(T)/T$ on $\mathrm{Fix}(T)$. It follows that if (Σ, T) is admissible (for flows, homeomorphisms or diffeomorphisms) in $\mathbb{R}^n$ with reversing symmetry group Γ, $(\Sigma_T, 1)$ is admissible in $\mathbb{R}^{n_T}$ with reversing symmetry group Γ_T, where $\Sigma_T := \Sigma/T$. However, this observation does not lead to an optimal result due to the presence of so-called *hidden symmetries* [8] which are elements γ of $\Gamma - N(T)$ with the property that $\gamma(\mathrm{Fix}(T)) \cap \mathrm{Fix}(T) \neq \emptyset$.

In order to determine the admissibility of a pair of subgroups (Σ, T) it suffices to take account of *hidden reflections*, which are elements $\gamma \in \Gamma - N(T)$ such that $\mathrm{Fix}(\gamma)$ intersects $\mathrm{Fix}(T)$ in a codimension one subspace. Following [6] we denote the set of $\gamma \in \Gamma$ for which $\mathrm{Fix}(\gamma)$ intersects $\mathrm{Fix}(T)$ in a codimension one subspace as K_T:

$$K_T := \{\gamma \in \Gamma \mid \mathrm{Fix}(\gamma) \text{ intersects } \mathrm{Fix}(T) \text{ in a codimension-one subspace } \} \tag{7}$$

Hidden reflections in Γ give rise to invariant subspaces within $\mathrm{Fix}(T)$ separating $\mathrm{Fix}(T)$ in two connected components.

Lemma 2.15 *Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a homeomorphism with reversing symmetry group Γ that is a finite subgroup of $O(n)$. Furthermore, let $T \leq \tilde{\Gamma}$ be an isotropy subgroup and $\rho \in \Gamma - \tilde{\Gamma}$ be a reversing symmetry that is not in $N(T)$. Suppose that $\mathrm{Fix}(\rho)$ intersects $\mathrm{Fix}(T)$ in a codimension one subspace, then this subspace is setwise invariant under f and pointwise invariant under f^2 (i.e. $f^2(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathrm{Fix}(\rho) \cap \mathrm{Fix}(T)$).*

Remark 2.16 *In [15] we obtained an analogous result in the context of flows: when the homeomorphism discussed in Lemma 2.15 is a flow f^t, then $f^t(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathrm{Fix}(\rho) \cap \mathrm{Fix}(T)$ and all $t \in \mathbb{R}$.*

Accordingly we define

$$\tilde{K}_T := K_T \cap ((\Gamma \cap N(T)^c) \cup (N(T) \cap \tilde{\Gamma})) \tag{8}$$

which is precisely the subgroup of K_T giving rise to invariant codimension one subspaces within $\mathrm{Fix}(T)$.

In analogy to (4)-(5) we define

$$L_T := \bigcup_{\tau \in K_T} \mathrm{Fix}(\tau), \quad \tilde{L}_T := \bigcup_{\tau \in \tilde{K}_T} \mathrm{Fix}(\tau). \tag{9}$$

¿From here it is straightforward to show that the results in the previous sections of this paper translate directly to the context of ω-limit sets in fixed point subspaces by simply replacing $\mathbb{R}^n$ by $\mathrm{Fix}(T)$, L by L_T, and $\tilde{L}$ by $\tilde{L}_T$.

3 Discussion

In this paper we have addressed certain aspects of the problem of admissibility of subgroups of reversing symmetry groups as symmetry groups of Liapunov stable ω-limit sets.

Following the ideas of [2, 6, 19] we have obtained necessary conditions for admissibility of general subgroups (Class I and Class II in Theorem 2.6), related to topological obstructions due to the presence of (reversing) reflection symmetries.

We have mainly focused on the situation in dimension $n \leq 2$. It should be noted that in higher dimensions more general results and techniques are expected to apply (in line with the results in the equivaraint category). In this respect, we would like to emphasize that the necessary conditions obtained in Theorem 2.6 are unlikely to be sufficient for determining the admissibility of subgroups in dimensions 3 and higher. We expect additional conditions that describe allowed permutations of connected components of attractors that are forced to be in different connected components of $\mathbb{R}^n - L$ (but in the same connected component of $\mathbb{R}^n - \tilde{L}$). We have not adressed this question in this paper (as it is not of great importance to the question of admissibility in dimensions 1 and 2).

Also, a more thorough understanding of the situation in the case of Class II subgroups when B is forced to be a reversing symmetry requires further analysis. In Lemma 4.4 we have shown that under certain conditions this leads to degenerate constructions with consequences for admissibility (cf. our discussion of the situation in dimension 2).

We intend to address systematic constructions of attractors in dimension $n \geq 3$ with more than one connected component in the near future.

We have left the problem of the admissibility of $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_{2m}(\mathbb{Z}_m)$ in $\Gamma(\tilde{\Gamma}) = \mathbb{Z}_{2m}(\mathbb{Z}_m)$ open. With the techniques employed in this paper we cannot decide on the (im)possibility of a connected attractor with these symmetry properties.

This open problem illustrates an important point. Periodic orbits and (KAM)-tori are Liapunov stable ω-limit sets that are relatively easy to construct in such a way that they admit reversing symmetries. However, the types of symmetry groups for which admissibility can be proven in this way are severely restricted. The general methods of constructions of Liapunov stable ω-limit sets employed in [6] appear not be adaptable to construct Liapunov stable ω-limit sets that are symmetric with respect to a reversing symmetry. Namely, in the constructions of [6] the ω-limit sets are automatically asymptotically stable. In the context of previous studies this property was usually welcomed. However, in the context of reversible (but also volume preserving and symplectic) attractors, such constructions are not useful. It might be that reversibility will impose additional restrictions beyond topological restrictions dealing with allowed permutations of connected components of attractors.

In our opinion, more consideration should be given to the notion of stability that is used. One could ask simply for the possible symmetries of ω-limit sets with no additional structure but to reflect the behavior of physical systems some form of stability is desirable. For Hamiltonian systems with no more than two degrees of freedom, e.g. in connection with the stability results implied by KAM-theory, cf. Moser [21], but also in the context of flows in $\mathbb{R}^4$ with a reversing symmetry fixing a two-dimensional hyperplane [16] the concept of Liapunov stability is reasonable. However, especially in higher dimensional systems the phenomenon of Arnol'd diffusion makes topological notions less adequate. Thom [24] has proposed the measure theoretic notion of a *vague attractor* as a Hamiltonian analog of the attractors of differentiable dynamics (cf. also [1, p.583]). Similar considerations exist in the work of Melbourne [17] concerning the notion of *essential asymptotic* stability. Another alternative would be to consider Nekhoroshev-stable ω-limit sets [22], which uses escape times instead of measure-theoretic or topological criteria. Importantly, Nekhoroshev-stability occurs persistently in nearly integrable Hamiltonian dynamical systems. For recent connections to KAM-theory see also [3, 20].

When the requirement for Liapunov stability is relaxed we foresee that constructions using heteroclinic connections may lead to more general results on the construction of symmetric ω-limit sets.

We have concentrated on the symmetry properties of general reversible homeomorphisms, diffeomorphisms, and flows. Similar questions could be asked about the symmetry properties of ω-limit sets in the more special class of symplectic (Hamiltonian) maps and flows. In this context it is worthwhile to note that the results on admissibility in dimension 2 also apply in the case of Hamiltonian flows and diffeomorphisms. Indeed, in $\mathbb{R}^2$ the constructions of periodic orbits in Section 2.2 can be done using solely (anti-)symplectic diffeomorphisms.

Another interesting line of investigation is the relation between symmetry and mixing properties in reversible (symplectic, Hamiltonian) systems. An elementary result in this direction is Theorem 2.2. Earlier work on equivariant systems has shown that topological mixing is incompatible with attractors having certain symmetries [19, 6]. As Hamiltonian systems have a natural invariant measure, relations between ergodic properties of the system and symmetry properties would be expected. Some spectral properties of ergodic reversible dynamical systems have been discussed recently in [10].

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4 Proofs:

The rest of this paper is mainly devoted to proving the results we presented above.

4.1 Restrictions

The aim of this section is to establish the restrictions on the symmetry groups of ω-limit sets as described in Theorem 2.6.

Reflection symmetries cause restrictions as their fixed point subspaces are setwise invariant under the dynamics. In contrast to this, fixed point subspaces of reversing symmetries need not be invariant under the dynamics. Recall from (5) that the subset $\tilde{L} \subset \mathbb{R}^n$ is defined to be the union of the fixed point subspaces of all reflections in $\tilde{\Gamma}$, where $\tilde{\Gamma}$ denotes the symmetry group asociated with the reversing symmetry group Γ. The restrictions will follow from generalizing results of [6, 19].

Proposition 4.1 *Assume*

(a) Y is a finite set and $|Y| = r$,

(b) $\mathbb{Z}_r$ acts transitively on Y,

(c) Σ has a normal subgroup $\tilde{\Sigma}$ of index 2 which acts fixed point freely on Y,

(d) the action of $\mathbb{Z}_r$ and $\tilde{\sigma}$ (where $\tilde{\sigma} \in \tilde{\Sigma}$) commute and $\rho a = a^{-1}\rho$ for all $a \in \mathbb{Z}_r$ and $\rho \in \Sigma - \tilde{\Sigma}$, then $\Sigma \simeq \mathbf{D}_k$ with k dividing r.

Proof. In [19, Proposition 4.3] it was shown that $\tilde{\Sigma} \simeq \mathbb{Z}_k$ with k dividing r. Now let $Y = \{y_0, \ldots, y_{r-1}\}$ with $y_i = a^i(y_0)$ and let $\rho(y_0) = y_m$ with $\rho \in \Sigma - \tilde{\Sigma}$. Then $\rho(y_i) = \rho a^i(y_0) = a^{-i}\rho(y_0) = a^{m-i}(y_0)$ and $\rho^2(y_i) = \rho a^{m-i}(y_0) = a^{i-m}\rho(y_0) = y_i$ for all $i = 0, \ldots, r-1$. Hence, because $\tilde{\Sigma}$ acts fixed point freely on Y and $\rho^2 \in \tilde{\Sigma}$ we obtain that $\rho^2 = \mathrm{id}$. This is the case for every $\rho \in \Sigma - \tilde{\Sigma}$. The proof is concluded by verifying the fact that Σ is generated by a $\tilde{\sigma}$ and ρ satisfying (only) the relations $\tilde{\sigma}^k = \rho^2 = (\rho\tilde{\sigma})^2 = \mathrm{id}$ and hence $\Sigma = \langle \tilde{\Sigma}, \rho \rangle \simeq \mathbf{D}_k$. □

Recall that if Δ is a subgroup of Γ then L_Δ denotes the union of reflection hyperplanes corresponding to group elements not contained in Δ, and $\tilde{\Delta} := \Delta \cap \tilde{\Gamma}$.

Proposition 4.2 *Suppose $f : I\!R^n \mapsto I\!R^n$ has reversing symmetry group Γ and a Liapunov stable ω-limit set A with symmetry $\Sigma \leq \Gamma$, then there exists a subgroup Δ of Σ such that*

(a) Δ fixes a connected component of $I\!R^n - L_\Delta$ that is contained in a connected component of $I\!R^n - \tilde{L}$.

(b) $\tilde{\Delta}$ is a normal subgroup of Σ and $\tilde{\Sigma}$.

(c) $\tilde{\Sigma}/\tilde{\Delta} \simeq \mathbb{Z}_k$, and
$\Sigma/\tilde{\Delta} \simeq \mathbf{D}_k$ when $\Sigma \neq \tilde{\Sigma}$.

Proof. We closely follow the proof of [19, Theorem 4.10, Lemma 2.1, Proposition 2.4 and Corollary 2.5]. Let L^* denote the reflection hyperplanes in L that do not intersect A. Then A is covered by finitely many disjoint connected components of $\mathbb{R}^n - \mathcal{P}_{L^*}$ (where $\mathcal{P}_{L^*} = \cup_{i=0}^{\infty} f^{-i} L^*$): $A \subset C_0 \cup \ldots \cup C_{n-1}$. Moreover, f cyclically permutes these components: $f(C_i) \subseteq C_{(i+1) \bmod n}$. Let $\Delta_i := \{\delta \in \Sigma \mid \delta(C_i) = C_i\}$. By equivariance, it then follows that $\tilde{\Delta}_i = \tilde{\Delta}_j =: \tilde{\Delta}$, and $\tilde{\Delta}$ is a normal subgroup of $\tilde{\Sigma}$ (because it is the kernel of the action of $\tilde{\Sigma}$ on $\{C_i\}$) and also of Σ (because $\tilde{\Sigma}$ is a normal subgroup of Σ). This proves (b). Moreover, let $A_i := A \cap C_i$, then $\tilde{\Sigma}/\tilde{\Delta}$ acts fixed point freely on the components $\{A_0, \ldots, A_{n-1}\}$. Hence, by Proposition 4.1 it follows that $\tilde{\Sigma}/\tilde{\Delta} \simeq \mathbb{Z}_k$ and when $\Sigma \neq \tilde{\Sigma}$ that $\Sigma/\tilde{\Delta} \simeq \mathbf{D}_k$, proving (c).

Because f fixes $\tilde{L}$, f also permutes components of $\mathbb{R}^n - \tilde{L}$, and moreover, A can intersect not more than two such components [18, 6]. Thus either Σ fixes a connected component of $\mathbb{R}^n - \tilde{L}$, or Σ contains a normal subgroup Σ' of index two that fixes a connected component of $\mathbb{R}^n - \tilde{L}$. In any case Σ thus possesses a subgroup Σ' fixing a connected component of $\mathbb{R}^n - \tilde{L}$. We take Δ to be the subgroup of Σ' described above. Thus Δ fixes a connected component of $\mathbb{R}^n - \tilde{L}$. The remaining point (a) follows from asserting that $L_\Delta \subseteq L^*$. Hence as Δ must fix a connected component of $\mathbb{R}^n - L^*$, the assertion implies that Δ also fixes a connected component of $\mathbb{R}^n - L_\Delta$.

To prove the assertion, following [19], we assume that $\mathrm{Fix}(\tau) \subset L_\Delta$, thus $\tau \notin \Delta$, and show that $\mathrm{Fix}(\tau) \in \mathrm{L}^*$. Suppose $\mathrm{Fix}(\tau) \notin L^*$, then $\mathrm{Fix}(\tau) \cap A \neq \emptyset$. However, in that case it follows from Theorem 2.1 that $\tau \in \Delta$. This contradicts our assumption that $\tau \notin \Delta$ and hence $L_\Delta \subseteq L^*$.

It should be noted that if Σ does not fix a connected component of $\mathbb{R}^n - \tilde{L}$, then k must be even in (c). □

Our proof of Theorem 2.6 uses Proposition 4.2 and Lemma 4.3.

Lemma 4.3 *Suppose that $f : I\!R^n \mapsto I\!R^n$ is a homeomorphism with reversing symmetry group Γ (which is a finite subgroup of $O(n)$) and that $\{C_i\}_{i=1}^p$ are the connected components of $I\!R^n - \tilde{L}$. Then there exists an involution $B \in C(\Gamma)$ such that $f(C_i) = B(C_i)$ for all $i = 1, \ldots, p$.*

Proof. The proof closely follows [6, §7]. First of all, f^2 fixes each connected component of $\mathbb{R}^n - \tilde{L}$. We let $\mathcal{R}$ denote the subgroup of $\tilde{\Gamma}$ generated by reflections in $\tilde{\Gamma}$. As shown in [6] each connected component C of $\mathbb{R}^n - \tilde{L}$ is a fundamental domain for the action of $\mathcal{R}$ on $\mathbb{R}^n$. Furthermore we may write $\mathbb{R}^n = V \oplus W$ where $V = V_1 \oplus ... \oplus V_p$ and $\mathcal{R} = R_1 \times ... \times R_p$ and R_j acts on V_j as a nontrivial irreducible finite reflection group. V and W are $\mathcal{R}$ invariant spaces and if C is a connected component of $\mathbb{R}^n - \tilde{L}$ then $C = F_1 \times ... \times F_p \times W$ where F_j is a fundamental domain for $R_j|_{V_j}$, $j = 1, ..., p$, and $\mathcal{R}$ acts trivially on W. Note that if $C = F_1 \times ... \times F_p \times W$ then $f(C) = e_1 F_1 \times ... e_p F_p \times W$ where $e_i \in \{-1, +1\}$. This follows from the observation of [6, §7] that if $J \subset \mathcal{R}$ is a maximal isotropy subgroup of $\mathcal{R}$ then $\mathrm{Fix}(J) \cap V$ is one-dimensional. We shall call $\mathrm{Fix}(J)$ an axis of symmetry for $\mathcal{R}$. Note that if V_j has dimension n_j then F_j intersects precisely n_j axes of symmetry. Furthermore if F_j and F_j' are fundamental domains for R_j acting on V_j and both $\bar{F}_j/0$ and $\bar{F}_j'/0$ intersect the axes of symmetry $A_1, ..., A_{n_j}$ then $F_j = \pm F_j'$. Now suppose that $f(C) = C' = F_1' \times ... \times F_p' \times W$. Since f is equivariant $f(A_i) = A_i$ for each i, where $\{A_i\}_{i=1}^{n_j}$ are axes of symmetry which intersect $\bar{F}_j/0$. Thus $\bar{F}_j'/0$ intersects each of the axes $A_1, ..., A_{n_j}$ and so $F_j = \pm F_j'$ as required. The subspaces $V_+ = \oplus_{e_i=1} V_i$ and $V_- = \oplus_{e_i=-1} V_i$ are shown in [6, §7] to be $\tilde{\Gamma}$ invariant. We will show that they are also Γ invariant. First note that the elements of Γ permute the connected components of $\mathbb{R}^n - \tilde{L}$. Each $\gamma \in \Gamma$ maps $\mathrm{Fix}(\tau)$ $(\tau \in \tilde{\Gamma})$ to $\mathrm{Fix}(\gamma\tau\gamma^{-1})$ $(\gamma\tau\gamma^{-1} \in \tilde{\Gamma})$, and so maps $\tilde{L}$ onto $\tilde{L}$.

Supppose A_i is an axis of symmetry for R_j acting on V_j. Then γA_i is an axis of symmetry for $\gamma R_j \gamma^{-1}$. If γ is a symmetry of f then f either preserves or reverses the orientation on A_i and γA_i together. Thus A_i and γA_i both lie in either V_+ or V_-. Since $f^{-1}(A_i) = f(A_i)$, it is also true in the case that γ is a reversing symmetry that A_i and γA_i both lie together in either V_+ or V_-.

Hence the axes of symmetry in V_+ and V_- are preserved by the action of Γ. Since the axes of symmetry span V_+ and V_-, V_+ and V_- are preserved by the action of Γ.

Thus we obtain $B \in C(\Gamma)$ by defining $B|_{V_+} = I|_{V_+}$, $B|_{V_-} = -I|_{V_-}$ and $B|_W = I|_W$. $\square$

In case the involution B constructed in the proof of Lemma 4.3 is forced to be a reversing symmetry, further restrictions on the map f may arise.

Lemma 4.4 *Consider the setting of Lemma 4.3. If the involution B as constructed in the proof of Lemma 4.3 is a reversing symmetry, and the subgroup $\mathcal{R}$ generated by the reflections in $\tilde{\Gamma}$ acts nontrivially on the whole of $\mathbb{R}^n$ (i.e. $\mathbb{R}^n = V$ and $W = \emptyset$) then the map f must be equal to B.*

Proof. Consider the setting of the proof of Lemma 4.3. Suppose that f is not an involution and the $\tilde{\Gamma}$-equivariant involution B is a reversing symmetry of f. Then it is readily verified that f may be written as $f = B \circ I$ where I is also a $\tilde{\Gamma}$-equivariant involution. Since $f(C) = B(C)$ for all connected components of $\mathbb{R}^n - \tilde{L}$ it follows that I must fix each such component. Now we consider the action of I to $\mathbb{R}^n = V$. Each component of $\mathbb{R}^n - \tilde{L}$ has the form $C = F_1 \times ... \times F_p$ and $I(C) = C$. Furthermore $I^2 = \mathrm{id}$. We will show that I must be the identity. To see this note that if Φ is an involution in $\mathbb{R}^{k+1}$ which fixes a k-dimensional hyperplane M pointwise, then either $\Phi = \mathrm{id}$ or Φ must permute the two connected components of $\mathbb{R}^{k+1} - M$.

Since I fixes the boundaries of all F_i setwise, and thus also fixes setwise the $(n-k)$-dimensional hyperplanes in which 2^k components meet without permuting them we see that

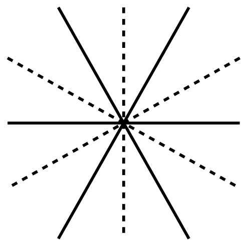

Figure 1: Schematic illustration of the action of $\Gamma = \mathbf{D}_6$ with $\tilde{\Gamma} = \mathbf{D}_3$. Solid lines represent reflection planes of symmetries and dashed lines represent reflection planes of reversing symmetries.

I is the identity. The fact that $I = \mathrm{id}$ implies that f must be identical to the involution B. $\square$

Example 4.5 *As an illustration of the point made in Lemma 4.4 let us consider the case where $\Gamma(\tilde{\Gamma}) = \mathbf{D}_6(\mathbf{D}_3)$. The action of Γ is schematically depicted in Fig. 1. Suppose that a homeomorphism $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ has Γ as a reversing symmetry group. Then if f permutes connected components of $\mathbb{R}^2 - \tilde{L}$ (which are in this case the sets separated by the solid lines in Fig. 1), there exists an involution $B \in C(\mathbf{D}_6)$ that induces the same permutation on the components. In this case there exists only one $\mathbf{D}_6$-equivariant involution $B \in C(\Gamma)$ that permutes components, namely $B = -\mathrm{id}$, which is a reversing symmetry ($-\mathrm{id} \in \Gamma - \tilde{\Gamma}$). Hence in this case it follows from Lemma 4.4 that every homeomorphism with reversing symmetry group Γ that permutes connected components of $\mathbb{R}^n - \tilde{L}$ must be equal to the involution $B = -\mathrm{id}$.*

Proof of Theorem 2.6. We know that an ω-limit set can intersect at most two components of $\mathbb{R}^n - \tilde{L}$. If an ω-limit set A is contained in a single component C_i of $\mathbb{R}^n - \tilde{L}$ and has symmetry Σ then Σ fixes A and C_i and thus by Proposition 4.2 Σ is of Class I.

Otherwise A intersects two connected components C_1,C_2 of $\mathbb{R}^n - \tilde{L}$. Following Melbourne [18], we define $\Sigma_i' := \{\sigma \in \Sigma;\ \sigma(C_i) = C_i\}$. Since Σ is a reversing symmetry group of f, and f interchanges C_1 and C_2 (so that both $f(C_1) = C_2$ and $f^{-1}(C_1) = C_2$) it follows that $\Sigma_1 = \Sigma_2 =: \Sigma'$. Σ' is a normal subgroup of Σ as it is the kernel of the action of Σ on $\{C_1, C_2\}$ and if $\Sigma' \neq \Sigma$ we find that $\Sigma/\Sigma' \simeq \mathbb{Z}_2$ (as an element of Σ either permutes the two components or fixes them). Since A has symmetry group Σ it is also fixed setwise by Σ' and so by Proposition 4.2 Σ' contains a subgroup Δ satisfying the definition of a Class I group. When $\Sigma \neq \Sigma'$ it follows from the fact that there must be an even number of connected components of $\mathbb{R}^n - \mathcal{P}_{L_\Delta}$ covering A so that $\tilde{\Sigma}/\tilde{\Delta} \simeq \mathbb{Z}_{2k}$, and $\Sigma/\tilde{\Delta} \simeq \mathbf{D}_{2k}$ when $\Sigma \neq \tilde{\Sigma}$. Lemma 4.3 asserts that $f(C) = B(C)$ with B being an involution in $C(\Gamma)$ that is not a reversing symmetry of f. Thus Σ satisfies all the properties required of a Class II subgroup. $\square$

4.2 Constructions of connected asymptotically stable symmetric ω-limit sets

In this section we establish the sufficiency results for strongly admissible subgroups (Theorem 2.13).

The proofs involve modifications to the corresponding proofs of [6] and we present in detail only the construction of Axiom A attractors for diffeomorphisms in dimension $n \geq 4$. The case $n = 3$ is handled in the same way.

Proof of Theorem 2.13. Let $\Sigma \leq \tilde{\Gamma}$. If a Σ-symmetric attractor is connected then it cannot intersect the fixed point subspace of a reflection symmetry. By Theorem 2.1 it also cannot intersect the fixed point subspace of a reversing symmetry, otherwise $\Sigma \not\leq \tilde{\Gamma}$. Hence A must be confined inside a connected component of $\mathbb{R}^n - L$.

Suppose now that Σ fixes a connected component of $\mathbb{R}^n - L$. Let $\tilde{\Gamma}$ be the symmetry group associated to Γ and let Σ be a subgroup of $\tilde{\Gamma}$. If $n \geq 4$ we may as in [6] construct a $\tilde{\Gamma}$-equivariant diffeomorphism with a Σ-symmetric Axiom A attractor A. This attractor has $|\tilde{\Gamma} : \Sigma|$ conjugate attractors. In the construction of f and A we may require that A has a Σ invariant neighbourhood U such that $f(U) = U$ and $f|_{\tilde{\Gamma}U^c} = \mathrm{id}|_{\tilde{\Gamma}U^c}$ (this follows from the construction of [6]). Furthermore we may require that $\gamma(U) \cap U = \emptyset$ for all $\gamma \in \Gamma - \Sigma$. The existence of such an open set U is guaranteed since Σ is admissible for Γ-equivariant diffeomorphisms by the work of [6].

Now take a Σ invariant neighbourhood $U' \supset U$ which also satisfies $\gamma(U') \cap U' = \emptyset$ for all $\gamma \in \Gamma - \Sigma$. We may smoothly isotope $f|_U$ to a diffeomorphism g which is the identity map on U'^c (since by construction in [6] we may take f Γ-equivariantly isotopic to the identity and also assume $f(U) = U$) and which has the properties $g_{|U} = f_{|U}$ and $g(U') = U'$.

Now we extend g Γ-equivariantly to a diffeomorphism h. Finally define a diffeomorphism F by $F(x) = h(x)$ if $x \in \tilde{\Gamma}(U')$, $F(x) = h^{-1}(x)$ if $x \in \rho\tilde{\Gamma}(U')$ (with $\rho \in \Gamma - \tilde{\Gamma}$) and $F(x) = g(x)$ on the complement of these two sets. Note that F has reversing symmetry group Γ and F has a Σ symmetric Axiom A attractor contained in U'. $\square$.

4.3 Constructions of attractors in dimension 1 and 2

In this section we discuss admissibility in $\mathbb{R}$ and $\mathbb{R}^2$. We will show that in most cases admissibility can be realized by the construction of symmetric periodic orbits. These constructions will form the proof of Theorem 2.7 (with reference to Table 3).

Before treating the constructions in $\mathbb{R}^2$, we first prove the results on admissibility in the one-dimensional case.

Proposition 4.6 Σ $(\tilde{\Sigma}) = \mathbb{Z}_2$ (1) *is admissible for homeomorphisms and diffeomorphisms in* $\mathbb{R}$ *with reversing symmetry group* Γ $(\tilde{\Gamma}) = \mathbb{Z}_2$ (1). *However, the only map* f *that realizes admissiblity for this subgroup is* $f = -\mathrm{id}$.

Proof. Γ is generated by a reversing symmetry $\rho = -\mathrm{id}$, such that $\tilde{\Gamma} = 1$. It is trivial to see that indeed the map $f = -\mathrm{id}$ has the required symmetry properties and Liapunov stable ω-limit sets with full symmetry (i.e. the orbits of period two).

We now show that this is the only map that realizes admissibility. We first recall that (bounded) ω-limit sets of homeomorphisms $\mathbb{R} \mapsto \mathbb{R}$ must be periodic orbits [5]. Hence if Σ $(\tilde{\Sigma}) = \mathbb{Z}_2$ (1) were admissible, then there would exist a $\mathbb{Z}_2$-symmetric periodic orbit. Such an orbit must intersect $\mathrm{Fix}(\rho) \cup \mathrm{Fix}(I)$ where $f = \rho \circ I$ and $\rho^2 = I^2 = \mathrm{id}$. We note that every continuous involution $I : \mathbb{R} \mapsto \mathbb{R}$ that is not equal to the identity transformation, fixes at most one point. In case $\mathrm{Fix}(\rho) = \mathrm{Fix}(I)$ the only possible symmetric ω-limit set is a fixed point in $\mathrm{Fix}(\rho)$. If $\mathrm{Fix}(\rho) \neq \mathrm{Fix}(I)$, however, it follows that the orbits intersecting

Γ $(\tilde{\Gamma})$	Σ $(\tilde{\Sigma})$	
$\mathbb{Z}_{2m}$ $(\mathbb{Z}_m)$,	$\mathbb{Z}_{2k}(\mathbb{Z}_k)$, $\mathbb{Z}_k(\mathbb{Z}_k)$	Class I
$\mathbf{D}_m$ $(\mathbb{Z}_m)$	$\mathbf{D}_k$ $(\mathbb{Z}_k)$, $\mathbb{Z}_k$ $(\mathbb{Z}_k)$	Class I
$\mathbf{D}_2$ $(\mathbf{D}_1)$	$\mathbf{D}_1$ (1), 1 (1)	Class I
	$\mathbf{D}_2$ $(\mathbf{D}_1)$, $\mathbf{D}_1$ $(\mathbf{D}_1)$, $\mathbb{Z}_2$ (1),	Class II
$\mathbf{D}_{2m}$ $(\mathbf{D}_m)$, $m \geq 2$ even	$\mathbf{D}_1$ (1), 1 (1)	Class I
	$\mathbb{Z}_2$ $(\mathbb{Z}_2)$	Class II
$\mathbf{D}_{2m}$ $(\mathbf{D}_m)$, $m \geq 3$ odd	$\mathbf{D}_1$ (1), 1 (1)	Class I
	$\mathbf{D}_1(\mathbf{D}_1)$, $\mathbb{Z}_2$ (1)	Class II

Table 5: The Class I and Class II subgroups $\Sigma(\tilde{\Sigma})$ of reversing symmetry groups $\Gamma(\tilde{\Gamma})$ that are finite subgroups of $O(2)$. Notation as in Table 3.

$\mathrm{Fix}(\rho)$ and $\mathrm{Fix}(I)$ cannot be periodic (as in that case f is homeomorphic to a translation). □

The rest of this section will be devoted to discussing the situation in dimension 2. ¿From Theorem 2.6 we know that if a subgroup is admissible it must be of Class I or of Class II. In Table 5 we list all Class I and Class II subgroups of finite subgroups of $O(2)$.

Lemma 4.7 *Let Γ $(\tilde{\Gamma}) = \mathbf{D}_m$ $(\mathbb{Z}_m)$, then all subgroups of Γ are admissible for homeomorphisms and diffeomorphisms in $\mathbb{R}^2$ with reversing symmetry group Γ.*

Proof. The proof of this Lemma uses the results we obtained on symmetric periodic orbits of flows (cf. Table 4 and [15]). If Γ $(\tilde{\Gamma}) = \mathbf{D}_m$ $(\mathbb{Z}_m)$, then we may construct a flow with reversing symmetry group Γ and a Γ-symmetric periodic orbit. By taking appropriate time-one maps, all subgroups of Γ can be realized. □

Lemma 4.8 *Let Γ $(\tilde{\Gamma}) = \mathbf{D}_{2m}$ $(\mathbf{D}_m)$, then $\mathbf{D}_1(1)$ and $1(1)$ are admissible Class I subgroups of Γ for homeomorphisms and diffeomorphisms in $\mathbb{R}^2$ with reversing symmetry group Γ.*

If $m \geq 3$ is odd, the only admissible Class II subgroup is $\mathbb{Z}_2(1)$, however the only map that realizes admissibility for this subgroup is $f = -\mathrm{id}$.

If $m = 1$, or $m \geq 2$ is even, then the only admissible Class II subgroup is $\mathbb{Z}_2(\mathbb{Z}_2)$.

For all admissible subgroups, admissibility can be realized by Liapunov stable periodic orbits.

Proof. When Γ $(\tilde{\Gamma}) = \mathbf{D}_{2m}$ $(\mathbf{D}_m)$, then the only nontrivial Class I subgroup is $\mathbf{D}_1$ (1). Because this subgroup is admissible for flows with reversing symmetry group Γ, admissibility is guaranteed by taking an appropriate time-one map of such a flow.

When $m \geq 2$ is even, then the Class II subgroup $\mathbb{Z}_2(\mathbb{Z}_2)$ is realizable by composing the time-one map of a flow that possesses a stationary point with trivial isotropy, with the involution $B = -\mathrm{id}$.

When $m \geq 3$ is odd, the only involution B in the centralizer $C(\Gamma)$ that interchanges the connected components of $\mathbb{R}^n - \tilde{L}$ is $B = -\mathrm{id}$, which is a reversing symmetry. Hence, by Lemma 4.4 we find that if f realizes admissibility for a Class II subgroup, f must be equal to $-\mathrm{id}$. This map has Liapunov stable periodic orbits with symmetry $\mathbb{Z}_2(\mathbb{Z}_2)$. $\mathbf{D}_1(\mathbf{D}_1)$ is not admissible.

The remaining case to be discussed is $\Gamma(\tilde{\Gamma}) = \mathbf{D}_2(\mathbf{D}_1)$. To establish the admissibility of $\mathbf{D}_1(\mathbf{D}_1)$ we compose the time-one map of a flow with reversing symmetry group Γ and a stationary point with trivial isotropy, with the reflection symmetry corresponding to the generator of $\mathbf{D}_1(\mathbf{D}_1)$. (This reflection is our choice for the involution $B \in C(\Gamma)$.)

There are two Class II subgroups left to consider. We will prove their admissibility using the same choice of B as above. The first subgroup we consider is $\Sigma\ (\tilde{\Sigma}) = \mathbf{D}_2\ (\mathbf{D}_1)$. We first construct a diffeomorphism g with reversing symmetry group Γ and a periodic orbit of period three inside a connected component of $\mathbb{R}^n - \tilde{L}$ that is symmetric with respect to the reversing reflection (this orbit has one point on the fixed set of the reversing reflection). Now we compose this map with the involution B. We obtain in this way a map with a periodic orbit of period 6 that has the desired symmetry properties.

The construction in case $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_2(1)$ is analogous. We first construct a diffeomorphism g with a $\mathbf{D}_1(1)$ symmetric period-two orbit that has no point on the fixed set of the reversing reflection. Then we compose this map with the reflection B. □

The remaining case to be discussed are the subgroups of $\Gamma\ (\tilde{\Gamma}) = \mathbb{Z}_{2m}\ (\mathbb{Z}_m)$. We will show that for most subgroups there are topological restrictions that prevent admissibility. For almost all other subgroups we can construct symmetric periodic orbits. The only subgroup for which our results are inconclusive is Γ itself.

We first establish a topological restriction on the symmetry groups of connected components of attractors that is due to the fact that we are in dimension 2.

Lemma 4.9 *Let A be a connected attractor of a homeomorphism in $\mathbb{R}^2$ with reversing symmetry group $\Gamma(\tilde{\Gamma}) = \mathbb{Z}_{2m}(\mathbb{Z}_m)$. If A is invariant under a nontrivial subgroup Σ of the reversing symmetry group Γ, then $\Sigma = \Gamma$.*

Proof. Consider a connected set A in $\mathbb{R}^2$ that is invariant under a nontrivial subgroup of $\mathbb{Z}_{2m}$. Then the image of this set under the generator ρ of $\mathbb{Z}_m$ will always intersect its preimage, i.e. $\rho(A) \cap A \neq \emptyset$. From Theorem 2.1 it follows that if A is a Liapunov stable ω-limit set then $\rho(A) = A$. □

Corollary 4.10 *If A is a Σ-invariant attractor of a homeomorphism in $\mathbb{R}^2$ with reversing symmetry group $\Gamma(\tilde{\Gamma}) = \mathbb{Z}_{2m}(\mathbb{Z}_m)$, where $\Sigma < \Gamma$ ($\Sigma \neq \Gamma$), then the connected components of A must have trivial symmetry.*

Proposition 4.11 $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_k(\mathbb{Z}_k)$ *is not admissible for homeomorphisms in $\mathbb{R}^2$ with reversing symmetry group $\Gamma(\tilde{\Gamma}) = \mathbb{Z}_{2m}(\mathbb{Z}_m)$ where $k > 2$ divides m.*

In case $k = 2$ and m is even, $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_2(\mathbb{Z}_2)$ is admissible.

Proof. From [19] we know that any attractor A can be covered by a finite number of disjoint compact sets C_i so that $A \subset C_0 \cup \ldots C_{r-1}$ and $f(C_i) \subseteq C_{i+1\ (\mathrm{mod}\ r)}$. When we define $A_i := A \cap C_i$ then we find that $f(A_i) = A_{i+1\ (\mathrm{mod}\ r)}$. By Lemma 4.10, the connected components of the attractor A must have trivial symmetry. Hence we may isolate sets of connected components of A by a Σ-invariant closed set S. Then by [19] we find that A is covered by a finite number of connected components C_i of $\mathbb{R}^n - \mathcal{P}_S$ (where $\mathcal{P}_S$ is the preimage set of S under f). When the symmetry of A is $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_k(\mathbb{Z}_k)$ (with k dividing m) we may choose $r \geq k$, such that for some $f^q(A_i) = A_{i+q \pmod r} = \rho^{pm/k}(A_i)$ for some positive integers q, and $p < k$ satisfying $\gcd(p, k) = 1$.

Thus, when $k > 2$ then f^q acts as a cyclic permutation on the sets A_i. To this action we can assign a sense of orientation. Importantly, the orientation of the cyclic action of f^{-q} on this set is opposite to that of f^q. Hence, we find that if ρ conjugates f^q to its inverse f^{-q}, ρ must be orientation reversing. However, ρ is an m-fold rotation and thus preserves orientation. This contradiction leads to the inadmissibility of $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_{2k}(\mathbb{Z}_k)$ with $k > 2$.

If $k = 2$, an attractor may be only covered by two connected components $C_0 \cup C_1$ and the contradiction we obtained when $k > 2$ fails to occur as we cannot assign an orientation to the cyclic permutation on only two sets. In fact, $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_2(\mathbb{Z}_2)$ is admissible when m is even. Namely, we may construct a diffeomorphism g with reversing symmetry group $\Gamma(\tilde{\Gamma}) = \mathbb{Z}_{2m}(\mathbb{Z}_m)$ and m even, that has a fixed point with trivial symmetry. If we compose this diffeomorphism with $-\mathrm{id}$, we obtain a diffeomorphism with the same symmetry properties and a $\mathbb{Z}_2$-symmetric periodic orbit. □

Proposition 4.12 *All subgroups* $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_{2k}(\mathbb{Z}_k)$ *with* $2 < k < m$ *are inadmissible for homeomorphisms in* $\mathbb{R}^2$ *with reversing symmetry group* $\Gamma(\tilde{\Gamma}) = \mathbb{Z}_{2m}(\mathbb{Z}_m)$ *(where* k *divides* m*).*

$\mathbb{Z}_2(1)$ *is an admissible subgroup when* m *is odd. However, when* $m > 1$*, the only map* f *that realizes admissibility is* $f = -\mathrm{id}$*.*

Proof. When $\Sigma \neq \Gamma$, it follows from Lemma 4.10 that the connected components of an attractor can only have trivial symmetry. Hence by Theorem 2.6 we find that $\Sigma \simeq \mathbf{D}_p$ for some p. This reduces the options to only one: $\Sigma(\tilde{\Sigma}) = \mathbb{Z}_2(1)$ (as $\mathbb{Z}_2 \simeq \mathbf{D}_1$).

We will now show that $\mathbb{Z}_2(1)$ is admissible, but that when $m > 1$ admissibility can only be realized by the map $f = -\mathrm{id}$.

Admissibility of $\mathbb{Z}_2(1)$ by the map $f = -\mathrm{id}$ is trivial. We will now show that this is the only homeomorphism realizing admissibility when $m > 1$.

We will prove the assertion by contradiction. Namely, we will show that when $f \neq -\mathrm{id}$, then f should be orientation reversing. However, on the other hand we will show that when f has the desired properties, f has to be orientation preserving.

We first argue why f must be orientation reversing. We write f as the composition $f = \rho^m \circ I$ (note that $\rho^m = -\mathrm{id}$). Because ρ^m is a reversing symmetry of f it follows that $I^2 = \mathrm{id}$. Now our attractor A can be covered by two disjoint compact sets C_0 and C_1 such that when $A_i := A \cap C_i$, then $f(A_0) = A_1 = \rho^m(A_0)$.

We first note that because $\tilde{\Gamma}$ is a nontrivial cyclic point group, the origin $\mathbf{0}$ is a fixed point of f, and hence also of I. Now consider a ρ-invariant line segment through $\mathbf{0}$ connecting A_0 to A_1. By considering the image of l under I, it is easily shown that I must be orientation reversing. Hence, because f is the composition of ρ and I, and ρ is orientation preserving, it follows that f must be orientation reversing.

Alternatively, we may decompose f in a different way. When we write $f = \rho \circ R^{-1}$, the fact that ρ is a reversing symmetry of f implies that $\rho^2 = R^2$ [12]. When $m > 1$, ρ^2 is an m-fold rotation in $\mathbb{R}^2$, hence so is R^2. It is now easily verified that hence R itself must be orientation preserving too. In turn this implies that f is orientation preserving. This concludes the proof of the fact that when $m > 1$ the only map that realizes admissibility of $\mathbb{Z}_2(1)$ is $f = -\mathrm{id}$.

When $m = 1$ the map f need not be orientation preserving, as f may just be the composition of $\rho = -\mathrm{id}$ and an orientation reversing involution I. Admissibility in this case can be obtained by a periodic orbit. We consider first the time-one map g of a flow with reversing symmetry group $\mathbf{D}_1(1)$ (!) with one stationary point $\mathbf{x}$ with isotropy $\mathbf{D}_1(1)$ away from the origin $\mathbf{0}$, and another stationary point at $-\mathbf{x}$. We compose this map with a linear reflection B, so that $B(\mathbf{x}) = -\mathbf{x}$. This construction can be done in such a way that the periodic orbit $(\mathbf{x}, -\mathbf{x})$ of the resulting map $f = B \circ g$ is Liapunov stable. Note that f is a diffeomorphism with reversing symmetry group $\mathbb{Z}_2(1)$, as required. For a systematic discussion of changes of reversing symmetry groups after composition, see [14]. $\square$

4.4 Symmetric attractors inside fixed point subspaces

Proof of Lemma 2.15 Let $T \leq \tilde{\Gamma}$, then for all $\mathbf{x} \in \mathrm{Fix}(T)$ we have $f(\mathbf{x}) \in \mathrm{Fix}(T)$. Using the fact that Γ is a finite subgroup of $O(n)$ we furthermore obtain that if $\mathrm{Fix}(\rho)$ intersects $\mathrm{Fix}(T)$ in a codimension one subspace and $\rho \notin N(T)$, then for all $\mathbf{x} \in \mathrm{Fix}(T) - \mathrm{Fix}(T) \cap \mathrm{Fix}(\rho)$, $\rho(\mathbf{x}) \notin \mathrm{Fix}(T)$.

Now consider some point $\mathbf{x} \in \mathrm{Fix}(\rho) \cap \mathrm{Fix}(T)$. For this point it follows that $f(\mathbf{x}) \in \mathrm{Fix}(T)$ and $f(\mathbf{x}) = \rho(f^{-1}(\mathbf{x}))$. As for all $\mathbf{x} \in \mathrm{Fix}(T) - \mathrm{Fix}(T) \cap \mathrm{Fix}(\rho)$, $\rho(\mathbf{x}) \notin \mathrm{Fix}(T)$, this implies that $f(\mathbf{x}) \in \mathrm{Fix}(\rho)$. In turn, this implies that $f(\mathbf{x}) = f^{-1}(\mathbf{x})$, so that for all $\mathbf{x}$ in this subspace $f^2(\mathbf{x}) = \mathbf{x}$. $\square$

Appendix

A Symmetric periodic orbits of flows (Corrigendum to [15])

In the context of flows in [15, Theorem 1.3] it was erroneously claimed that $\tilde{\Sigma}$ must be a maximal cyclic subgroup of $\tilde{\Gamma}$ whenever Σ is an admissible subgroup that does not fix a connected component of $\mathbb{R}^n - L$. This is only the case in $\mathbb{R}^2$.

The general situation is correctly described below.

Theorem A.1 *Let Γ be a reversing symmetry group which is a finite subgroup of $O(n)$ ($n \geq 3$), and Σ be a subgroup of Γ. Then there exists a flow $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$ with reversing symmetry group Γ with a Σ-symmetric periodic orbit (with trivial instantaneous symmetries), if and only if*

(i) when $\Sigma = \tilde{\Sigma}$, then $\Sigma \simeq \mathbb{Z}_k$ fixes a connected component of $\mathbb{R}^n - L$.

(ii) when $\Sigma \neq \tilde{\Sigma}$, then $\tilde{\Sigma} \simeq \mathbb{Z}_k$ and $\Sigma \simeq \mathbf{D}_k$, and $\Sigma = \langle \rho_1, \rho_2 \rangle$ with

- $\rho_i \in \Sigma - \tilde{\Sigma}$, $\rho_i^2 = \mathrm{id}$ $(i = 1, 2)$,
- $\exists$ *connected component C of $\mathbb{R}^n - L$, such that for $i = 1, 2$,* $\dim(\mathrm{Fix}(\rho_i) \cap \overline{C}) \geq 1$,
- $\mathrm{Fix}(\rho_i) \not\subseteq \mathrm{Fix}(\gamma)$ *for all $\gamma \in \tilde{\Gamma} - \{\mathrm{id}\}$ $(i = 1, 2)$.*

Proof. The proof of this theorem follows the arguments set out in [15]. Note in particular that condition (iii) is related to the fact that every periodic orbit that is symmetric with respect to a reversing symmetry ρ, must intersect $\mathrm{Fix}(\rho)$ in precisely two points. In that case also the reversing involutory generators of Σ must have fixed point subspaces that have a nonempty intersection with the closure of a connected component of $\mathbb{R}^n$. □

If Σ does not fix a connected component of $\mathbb{R}^n - L$ then ω-limit sets with symmetry Σ must be periodic orbits.

Theorem A.2 *Let Γ be a reversing symmetry group that is a finite subgroup of $O(n)$, and $\Sigma \leq \Gamma$. If Σ does not fix a connected component of $\mathbb{R}^n - L$, then Σ is admissible for flows in $\mathbb{R}^n$ with reversing symmetry group Γ if and only if Σ satisfies the conditions (i), (ii), and (iii) of Theorem A.1. The ω-limit sets that realize admissibility must be periodic orbits.*

Proof. The ω-limit sets that realize admissibility must intersect a reversing reflection hyperplane twice. Hence these ω-limit sets must be periodic orbits. □

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