Detection of symmetry of attractors from observations.
Part I: Theory

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Abstract

Barany *et al.* [Physica D **67**:66-87, 1993] propose a method for determining the symmetries of attractors of equivariant systems by averaging certain classes of equivariant maps. We use an idea in Barany *et al.* to re-cast definitions of symmetry detectives assuming that we only have access to (equivariant) observations from the system. Detecting from observations allows one to perform averaging in spaces that may have much lower dimension than the phase space. This paper generalises and develops their suggestion.

Among the generalisations we consider are the use of non-polynomial detectives, and we show using the notion of "prevalence" of Hunt *et al.* that our detectives and the detectives of [5, 10, 15] give the correct symmetry of attractors "almost certainly" in a measure-theoretic sense. We show that detectives can persistently give incorrect symmetries at isolated points in parametrised systems and discuss how to overcome this. We show how one can find the symmetry of an attractor from examination of a Poincaré section.

In part II of this article, Ashwin and Tomes apply these results to find symmetries of attractors in a physical system of four coupled electronic oscillators with $\mathbf{S}_4$ symmetry.

1 Introduction

The recent development of symmetry detectives [5] (see also [10, 15, 23, 13, 21]) makes it theoretically possible to detect symmetries of attractors. However, it is

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difficult to apply the definitions and theorems of these papers to physical systems because one usually does not have access to the phase space, but rather to observations taken from the phase space. The purpose of this paper is to re-cast the definition of detectives along lines suggested by [5, Section 10] so that detective ideas can be applied rigorously not just to numerical or theoretical examples but also to practical experiments.

Suppose we have a system with a symmetry group Γ. We define a notion of detective that will give the correct answers for *generic symmetric observations* from the phase space of the system M into a low dimension space S. The detectives are then maps from S into another representation space W of the symmetry group; note that we have introduced an intermediate space S and consider a composition of maps:

$$M \xrightarrow{\psi} S \xrightarrow{\phi} W.$$

Roughly speaking, the map ϕ will be called a *detective* if the isotropy of the average of $\phi \circ \psi$ gives the symmetry of an attractor of a dynamical system in M for "almost all" symmetric observations ψ.

We address the following problems: How do we rigorously characterise detectives? How do we characterise detectives that will work not only for one attractor but also for a d-parameter family of attractors? How do we detect symmetries from a Poincaré section?

The paper is organised as follows. In Section 1.1 we discuss instantaneous and setwise symmetries of attractors, establish notation and give a brief discussion of restrictions on possible symmetries for attractors. Section 2 gives definitions of detectives (Definition 2.6) in our setting and proves some results about a large class of detectives (Theorem 2.8). In Section 3 we present results pertaining to the prevalence of detectives, and in particular we show that detectives correctly determine the symmetries of attractors "almost certainly". In Section 4 we turn our attention to d-parameter systems where detectives may fail to give the correct symmetries at points of higher codimension in the parameter space. We give necessary conditions that a detective works for a d-parameter system. Section 5 shows that we may correctly determine the symmetries of attractors from an appropriately chosen Poincare section. We conclude with a discussion in Section 6. For completeness we include an appendix covering some fundamental ideas necessary to apply prevalence results.

1.1 Symmetries of attractors: Notation

Attractors in symmetric systems often have a symmetry that is a subsymmetry of the system. Local bifurcation of solutions with symmetry typically give rise to solutions with lower symmetry through a process called *spontaneous symmetry breaking*. Ideas of local bifurcations work well when one restricts to discussion of fixed points or periodic orbits, but generally fail to say much about more complicated behaviour.

Chossat and Golubitsky [9] hypothesised the existence of symmetry increasing bifurcations of chaotic attractors and studied some examples for $\mathbf{D}_n$ symmetric maps of the plane. Since then, work has been done to classify admissible symmetries of attractors [19, 3, 12] and some progress has been made on discovering mechanisms by which this symmetry can change [11, 2].

We shall consider a dynamical system defined by a map $x_{n+1} = f(x_n)$ or an ordinary differential equation $\dot{x} = F(x)$ on $\mathbf{R}^n$. The defining functions are assumed to be equivariant (i.e. $\gamma f = f\gamma$ for all $\gamma \in \Gamma$, similarly for F) under an action of a finite group Γ on M. We define an *attractor* A to be a Liapunov stable ω-limit set.

We define a metric d_m on compact subsets of $\mathbf{R}^n$ such that sets close in this metric are close both pointwise and in the sense of Lebesgue measure. Given two (Borel measurable) bounded subsets A and B of $\mathbf{R}^n$ we define $d_l(A,B) = \ell(A \triangle B)$ where Δ is the setwise symmetric difference. The metric d_m is defined by

$$d_m(A,B) = d_H(A,B) + d_l(A,B).$$

with d_H the usual Hausdorff metric.

We can use these metrics to define symmetries of sets (which will usually be attractors). The usual definition of the *symmetry on average* is the subgroup

$$\Sigma(A) \equiv \{\sigma \in \Gamma \ : \ d_H(\sigma A, A) = 0\}$$

An important subgroup of $\Sigma(A)$ is

$$T(A) \equiv \{\sigma \in \Gamma \ : \ \sigma x = x \ \text{ for all } x \in A\},$$

the *instantaneous symmetry* that fixes all points of A. As noted in Melbourne *et al.* [19], $T(A)$ is a normal subgroup of $\Sigma(A)$ and is an isotropy subgroup of the action corresponding to the intersection of the isotropies of points in A.

Other notation For a subgroup G acting on X we define the fixed point subspace of G to be $\mathrm{Fix}_X(G) = \{x \in X \ : \ \sigma x = x \text{ for all } \sigma \in G\}$. We write $\mathrm{Fix}(G)$ if the particular space X is clear from the context.

For a single point $y \in M$ the isotropy of y is $\Sigma(y) \equiv \Sigma(\{y\}) \equiv T(\{y\})$. For a vector space M write

$$\mathcal{A}(M) = \{ \text{ compact } A \subset M \ : \ \text{ either } \gamma A \cap A = \emptyset \text{ or } \gamma A = A \text{ for } \gamma \in \Gamma\}.$$

Note that Liapunov stable attractors are sets in $\mathcal{A}(M)$ [19, Proposition 4.8]. In addition, define

$$\mathcal{B}(M) = \{A \in \mathcal{A}(M) : \ \text{ points with isotropy } T(A) \text{ are dense}\}.$$

If $A \in \mathcal{A}(M)$ is an attractor with a dense orbit, then $A \in \mathcal{B}(M)$. Finally, $C^k_\Gamma(M,S)$ denotes the space of Γ-equivariant maps from M to S which are k times continuously differentiable ($k \geq 1$).

1.2 Attractors with nontrivial instantaneous symmetries

Suppose that Γ acts faithfully on the space V. To take proper account of all symmetries of a subset $A \subset V$ it is necessary to consider those that may be setwise symmetries as opposed to instantaneous symmetries, along the lines discussed in [15]. For example, suppose that $T(A) = G$ is non-trivial; then there may be symmetries of A that are not contained in G but are contained in the *normaliser* $N_\Gamma(G) = \Sigma(\mathrm{Fix}_V(G))$ of G in Γ. For each subgroup $G \subset \Gamma$ we define

$$G' \equiv N_\Gamma(G)/G.$$

When G' is trivial, A cannot have any additional average or setwise symmetries and thus detecting further symmetries beyond instantaneous symmetries is unnecessary. Therefore we make the following definition.

$$G(V) = \{G \subset \Gamma : G \text{ is an isotropy subgroup such that } G' \neq 1\}$$

Note that 1 (the trivial group) is always in $G(V)$ if V itself is non-trivial. We may sometimes assume that there exists an SBR (Sinai-Bowen-Ruelle) measure ρ_{SBR} on A; that is, an ergodic invariant measure that is generic for Lebesgue a.e. point in some neighbourhood U of A. We define an SBR attractor to be an attractor A with a dense orbit and an SBR measure. For further discussion see [6, 7, 10].

With a suitable definition of attractor, one can characterise the permissible symmetries, i.e. those symmetries of attractors that can be realised by a Γ-equivariant dynamical system. There are more stringent restrictions if we allow the dynamical system to be a flow or a diffeomorphism rather than just a map. The two cases are described in the papers [19, 3, 12]. Effectively the only restrictions are imposed by the existence of reflection hyperplanes of the group action, i.e. codimension one surfaces fixed by some subgroup. For the system of oscillators investigated in Part II we note that there are no such invariant hypersurfaces and so there are no *a priori* restrictions on possible average symmetries of attractors in this system.

2 Detecting the symmetry of attractors

Barany *et al.* [5] considered the following question: Given a trajectory from an equivariant dynamical system that converges to an attractor, how can one detect the symmetry of the attractor? They introduced the idea of a *detective*, a (vector-valued) equivariant function that when averaged over the attractor enables one to read off the symmetry. The theory of detectives has been generalised and discussed by several workers since; see [10, 15, 23, 8].

Previous work has mostly concentrated on cases where one has full access to the phase space and only polynomial detectives have been used. By introducing an intermediate observation space (suggested by Barany *et al.* [5, Section 10]) one can overcome the first restriction, whilst adapting an idea of Tchistiakov one can overcome the second.

2.1 Equivariant observables

King and Stewart [18] have considered the problem of reconstructing the phase space of a symmetric system and prove an equivariant version of the Takens embedding theorem [22]. They show that in order to reconstruct the dynamics one must consider equivariant observables that carry a 'complicated enough' representation of the group.

More precisely the group action on S must satisfy the representation theoretic condition that M is *subordinate* to S. This means that (a) S contains an isomorphic copy of every irreducible representation of any isotropy type of points in M and (b) every orbit type in M embeds equivariantly into $S^t \setminus \{0\}$ for some t. Note that (b) corresponds to a statement about the global geometry of M.

In this paper we shall assume that the phase space $M = \mathbf{R}^m$ and the observation space $S = \mathbf{R}^n$ for some m and n, with Γ a finite group acting orthogonally.

We do not consider phase-space reconstruction but instead merely wish to find observables such that we can detect the symmetries of the attractors. To this end, we consider equivariant observables that allow us to distinguish all isotropy types of the action of Γ on M. The following is a weaker equivalence than *lattice equivalent* considered in [5, Defn 4.2]; we do not require that the fixed point subspaces are isomorphic, just that they are in 1-1 correspondance.

Definition 2.1 *We say two orthogonal representations of Γ, W_1 and W_2, are* isotropy equivalent *if H is an isotropy subgroup for the action of Γ on W_1 if and only if it is an isotropy subgroup for the action on W_2.*

In order to distinguish all possible subgroups of a particular symmetry group we need to consider representations that satisfy the following definition:

Definition 2.2 *[5, equivalent to Defn 4.1] An orthogonal representation W of Γ is a* distinguishing representation *if all subgroups of Γ are isotropy subgroups.*

In [5, Theorem 4.3] it is shown that there exist distinguishing representations, namely W that contain all nontrivial irreducible representations of Γ at least once. As in [15] we will prove that the notion of detective we introduce will correctly determine the symmetries of attractors regardless of their instantaneous symmetries. We shall make use of the following result which is a simplification of [15, Lemma 2.3].

Lemma 2.3 *Suppose that W is a distinguishing representation for Γ. Then for any G, the action of $G' = \Sigma(Fix_W(G))/G$ on $Fix_W(G)$ is a distingushing representation for G'.*

Proof Consider any two subgroups $H_i' = H_i/G$ for $i = 1, 2$ of G'. Assume that $\mathrm{Fix}_W(H_1) \cap \mathrm{Fix}_W(G) = \mathrm{Fix}_W(H_2) \cap \mathrm{Fix}_W(G)$. Note that H_i satisfies $\Gamma \geq H_i \geq G$ and so $\mathrm{Fix}_W(H_i) \subset \mathrm{Fix}_W(G)$. Therefore we have $\mathrm{Fix}_W(H_1) = \mathrm{Fix}_W(H_2)$, contradicting the assumption that W is a distinguishing representation.

□

2.2 Relating $\Sigma(A)$ and $\Sigma(\psi(A))$

As the following example shows, given an arbitrary compact set A and observable ψ we can get the wrong answer for an open set of observations close to ψ, even if ψ works for other sets A!

Example: A **and** ψ **such that all** ψ' **close to** ψ **satisfy** $\Sigma(\psi'(A)) \neq \Sigma(A)$
We take $M = S = \mathbf{R}$ and consider $A = [-1.1, 1]$, $\psi = \sin 2\pi z$. Then it is clear that $\psi(A) = [-1, 1]$ and since the extrema of ψ are attained inside A, any small enough perturbation of ψ will still have that $\psi'(A)$ has $\mathbf{Z}_2$ symmetry, even though A does not.

For attractors in $\mathcal{B}(M)$, the next proposition shows that for a generic set of equivariant observables taking values in a large enough representation of Γ we have $\Sigma(\psi(A)) = \Sigma(A)$.

Proposition 2.4 *For any $A \in \mathcal{B}(M)$ and any S isotropy equivalent to M there is an open dense set of observations $\psi \in C^k_\Gamma(M, S)$ such that $\Sigma(\psi(A)) = \Sigma(A)$.*

Proof Since ψ is Γ-equivariant we have $\Sigma(A) \subset \Sigma(\psi(A))$. We will show that for each $\rho \in \Gamma - \Sigma(A)$ there is an open dense set X_ρ of observations such that $\rho\psi(A) \neq \psi(A)$. The set $\cap_{(\rho\in\Gamma-\Sigma(A))} X_\rho$ is then an open dense set of observations in $C^k_\Gamma(M, S)$ such that $\Sigma(\psi(A)) = \Sigma(A)$. Openness is clear because $d_H(\psi(A), \rho\psi(A))$ varies continuously with ψ. Now let $\rho \in \Gamma - \Sigma$ and suppose there exists $\psi \in C^k_\Gamma(M, S)$ such that $\rho\psi(A) = \psi(A)$. We will show that given $\epsilon > 0$ there exists an observation ψ_ϵ in $C^k_\Gamma(M, S)$ which is within ϵ of ψ in the C^k topology such that $\rho\psi_\epsilon(A) \neq \psi(A)$.

We first claim that there is a unit vector v with isotropy $T(A)$, and an $a \in A$ with isotropy $T(A)$ such that $(\psi(a) + rv) \notin \psi(A)$ for all $r > 0$.

To prove the claim, pick v such that $(\psi(a)+rv)$ has isotropy $T(A)$ for all $r > 0$. This is possible because the set of points with isotropy $T(A)$ is a finite union of convex cones and $\psi(a)$ is in the closure of these cones. Because $\psi(A)$ is closed, we can find $r_0 \geq 0$ such that $\psi(a) + r_0 v \in \psi(A)$ but $(\psi(a) + (r_0 + r)v) \notin \psi(A)$ for all $r > 0$. Since $\psi(a) + r_0 v$ has isotropy $T(A)$ it must be the image of a point with isotropy contained in $T(A)$.

Given such an a and v choose $\delta > 0$ such that $B_\delta(\rho(a))$ intersects A if and only if $\rho \in \Sigma(A)$ (we can do this because $A \in \mathcal{B}(M)$ and Γ is finite). Define $\eta \in C^k(M, \mathbf{R})$ supported on $B_\delta(a)$ with $\eta(a) = 1$ and define $\psi_\epsilon \in C^k_\Gamma(M, S)$ by

$$\psi_\epsilon(x) = \psi(x) + \epsilon \sum_{\gamma\in\Gamma} \gamma v \eta(\gamma^{-1}x).$$

Thus $\|\psi_\epsilon - \psi\|_{C^k} < \epsilon|T(A)|\,\|\eta\|_{C^k}$ which can be taken arbitrarily small (the C^k norm of η depends upon δ but is independent of ϵ). Note also that $\sum_{\gamma\in\Gamma} \gamma v\eta(\gamma^{-1}a) = v$. If $\rho \notin \Sigma(A)$ and $\rho\psi(A) = \psi(A)$ then

$$\rho\psi_\epsilon(A) = \psi_\epsilon(\rho A) = \psi(\rho A) = \psi(A)$$

Thus for any $\epsilon > 0$ we have

$$\min_{x \in \psi_\epsilon(A)} d(\rho(\psi(a) + \epsilon v), x) > 0$$

and so $d_H(\rho\psi_\epsilon(A), \psi_\epsilon(A)) > 0$. This implies that there are ψ_ϵ arbitrarily close to ψ (in the Whitney C^k topology) such that $\Sigma(\psi_\epsilon(A)) = \Sigma(A)$.
□

Remark 2.5 *If A is an attractor with a dense orbit then this orbit must have isotropy $T(A)$ and so $A \in \mathcal{B}(M)$. However, there are Liapunov stable invariant sets A (for example heteroclinic cycles) that can be attracting and even robust in systems with symmetries but which possess no points with isotropy $T(A)$.*

2.3 Detection from equivariant observables

Our definition of detectives involves composing an observable ψ with a detective ϕ in the following manner

$$M \xrightarrow{\psi} S \xrightarrow{\phi} W.$$

Note that if $S = M$ and ψ is required to be a diffeomorphism then we have the definition of Barany *et al.*. More precisely, they define the observable ϕ to be a detective if for each subset $A \in \mathcal{A}(M)$ (of positive Lebesgue measure) there is a residual set of diffeomorphisms $\psi \in \mathrm{Diff}^k_\Gamma(\mathbf{R}^n)$ (the space of Γ-equivariant k times continuously differentiable diffeomorphisms of $\mathbf{R}^n$) such that

$$\int_A \phi(\psi(y)) d\ell(y)$$

has symmetry $\Sigma(A)$.

Two methods of averaging are proposed by Barany *et al.*. One method (see also [10, 15]) involves taking the time average of an equivariant observable along a trajectory. This method can be shown to work if we assume there exists an SBR measure supported on the attractor (see [10, 15]). Numerical evidence suggests that this assumption is often justified; see the discussion in [10]. The convergence of this method may be very slow, but it has the advantage that it requires little computer memory to be practically implemented.

The other method involves "thickening" the attractor so that the resulting set has positive Lebesgue measure, and then performing numerical integration of an observable over this thickened attractor with respect to Lebesgue measure. It is clear that this method has severe limitations if the dimension of the phase space is large. In practise, it is impossible to perform this averaging method exactly; usually coarse-graining of the underlying space has been used (see Section 2.5).

With reference to the first method, the *ergodic average* of ϕ over a sequence $\{x_i \ : \ i \in N\}$ is defined to be

$$K^E_\phi(\{x_i\}) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \phi(x_j).$$

If A is an attractor in M supporting an SBR measure ρ_{SBR} we say that A is an SBR attractor and define

$$K^E_{\phi,\psi}(A) = \int_A \phi(\psi(x))\, d\rho_{SBR}(x).$$

By the definition of an SBR measure, for a positive (Lebesgue) measure set of initial conditions x we have

$$K^E_\phi(\{\psi(x), \psi(f(x)), \psi(f^2(x)), \cdots, \}) = K^E_{\phi,\psi}(A).$$

2.4 Averaging the observed attractor

The *integrated observed* method of averaging involves taking an image of the attractor in an observation space and then averaging over Lebesgue measure on a fixed point subspace in this observation space:

$$K^I_\phi(\psi(A)) = \int_{\psi(A)} \phi(x)\, d\ell(x)$$

where ℓ is Lebesgue measure [1] on $\mathrm{Fix}_S(T(A))$.

As noted by Barany *et al.* it is typical that A may have zero Lebesgue measure, for example if A is a periodic orbit. In this case $K^I(\psi(A)) = 0$ and so it is necessary to equivariantly "thicken" the attractor while keeping the symmetry constant to have a chance of detecting the correct symmetry.

We take the image $\psi(A)$ of the attractor in the observation space S and consider sets B with the same symmetry as A that are close both pointwise and measurewise. We then compute $K^I_\phi(B)$. The advantage of this method of averaging is that one need only integrate in relatively low dimensional spaces where the dimension is dependent upon the group representation. Even partial differential equations with infinite dimensional phase spaces can be considered using this method.

We can now define our detectives:

Definition 2.6 *A* detective *for M is a Γ-equivariant map $\phi : S \to W$ between two representations of Γ such that for all $k \in \mathbf{N}$ we have:*

(a) ϕ is an SBR detective *if for all SBR attractors $A \subset M$ there is an open dense set of observations $\psi \in C^k_\Gamma(M, S)$ such that*

$$\Sigma(K^E_{\phi,\psi}(A)) = \Sigma(A).$$

(b) ϕ is an integrated observed detective *if for all compact sets $C \subset S$ there is an open dense set in a neighbourhood (in the d_m-topology) $\mathcal{N}$ of C such that if $B \in \mathcal{N}$ and $\Sigma(B) = \Sigma(C)$ then*

$$\Sigma(K^I_\phi(B)) = \Sigma(C).$$

[1] Note that we must know $T(A)$ in order to know on which subspace we define the Lebesgue measure ℓ. Practically this is not a problem because for generic ψ we have $T(A) = T(\psi(A))$ and the latter is easily measured.

The open dense set of observations will be dependent on the attractor $A \in \mathcal{A}(M)$ and the map ϕ.

Remark 2.7 *By applying Proposition 2.4 for $A \in \mathcal{B}(A)$, (b) above implies that generically an integrated observed detective will give the correct symmetry of the underlying set in M. Note that we do not require that $C \in \mathcal{A}(S)$ to be satisfied or that the boundary is piecewise smooth for (b) and in this sense our definition is slightly weaker than that of [5].*

We now state a theorem generalising [5, Theorem 5.2] to give a large class of continuously differentiable detectives. We do not require the detectives to be polynomials or anything more than just continuously differentiable.

Theorem 2.8 *Let $\phi \in C^1_\Gamma(S, W)$ with S isotropy equivalent to M and W a distinguishing representation. Suppose that for each isotropy subgroup G for M (or equivalently S) and all neighbourhoods $N \in Fix_S(G)$ we have*

$$Span\{D_x\phi v : x \in N, v \in Fix_S(G)\} = Fix_W(G). \tag{2.1}$$

Then ϕ is a detective.

We relegate the proof to Appendix B. For polynomial equivariant maps, this theorem is equivalent to that of [5, 10, 15]:

Proposition 2.9 *Suppose that $\phi : S \to W$ is a Γ-equivariant polynomial map into a distinguishing representation W. Write $Fix_W(G) = \bigoplus W'_i$ as the isotypic decomposition of the action of $G' = \Sigma(Fix_W(G))/G$ on $Fix_W(G)$ and π_i the orthogonal projection onto W'_i. The following are equivalent:*

(a) For each $G \subset G(S)$ and each i the projection $\pi_i \circ \phi(Fix_S(G)) \subset W'_i$ is non-zero.

(b) For each neighbourhood $N \subset Fix^S(G)$ we have

$$Span\{D_x\phi v : x \in N, v \in Fix_S(G)\} = Fix_W(G).$$

Remark 2.10 *This implies that on restricting to* polynomial *equivariant observables, our sufficiency conditions for an equivariant observable to be a detective are equivalent to those given in [15].*

Proof (of Proposition 2.9) Suppose that for some i, $\pi_i \circ \phi$ is identically zero. Then the linearization $\pi_i D_x\phi$ is identically zero for every $x \in S$, a contradiction to condition (b). Thus condition (b) implies condition (a). Conversely, suppose that

$$\mathrm{Span}\{D_x\phi v : x \in N, v \in \mathrm{Fix}_S(G)\} \neq \mathrm{Fix}_W(G)$$

for some neighbourhood $N \subset \mathrm{Fix}_S(G)$. If $\phi : \mathbf{R}^n \to \mathbf{R}^m$ is a polynomial (smooth map) and if the images $D\phi_x v$ for all $x \in N, v \in \mathrm{Fix}_S(G)$ lie in a proper subspace of $\mathbf{R}^m$ then modulo a fixed constant vector the image of $\phi : S \to W$ ($\phi : N \to W$) also lies in that subspace. If ϕ is equivariant then the fixed constant vector must be the zero vector. We also note that $\pi_i D\phi_x = D\pi_i\phi_x$ so that if $D\phi_x$ is not onto then $D\pi_i\phi_x$ is contained strictly in a subspace of W_i for some i and hence (as W_i is irreducible and $\pi_i\phi$ is equivariant) $\pi_i\phi = 0$. Thus condition (b) implies condition (a).

□

Using Theorem 2.8 and Proposition 2.9 gives the following result,

Corollary 2.11 *Let Γ be a finite group acting on V and let W be a distinguishing representation. Define P_k to be the space of Γ-equivariant polynomial mappings of S to W of degree at most k. We give P_k its usual topology. In light of the theorem above and [15, Theorem 1.2] we have that for each sufficiently large k, there exists an open dense subset $\mathcal{D} \subset P_k$ such that each $\phi \in \mathcal{D}$ is a detective. Moreover, for any $k \geq 1$ there exists an open dense subset of $C^k_\Gamma(S, W)$ that are detectives.*

2.5 Discretised detectives

The method of thickening an attractor (either in the original phase space or thickening its image in some observation space) and then averaging over Lebesgue measure on this space cannot be done exactly. In general we approximate it numerically.

A natural way to do this (and the one used, for example in [5] and Part II of this paper) is to take a discrete lattice whose ϵ-neighbourhood covers the whole space and (piecewise continuously) project the attractor onto this grid. A great advantage of this to thickening the attractor is that if the grid is chosen to respect the invariant subspaces one does not have to worry about first measuring $T(A)$ and then thickening in $\mathrm{Fix}_S(T(A))$. However, a sensible definition for a *discretised detective* that enables results to be rigorously interpreted is still elusive.

3 Prevalence results

Because a topologically generic set can have small measure and, conversely, a set of large measure may be non-generic, it is desirable that our detectives give the correct answer not only for a topological (generic) large set of observations but also in a measure-theoretic sense.

The measure theoretic notion of prevalence [17] was developed to enable one to talk of a property holding "almost everywhere" on infinite dimensional vector spaces- a generalisation of full Lebesgue measure to an infinite dimensional setting. For further details on the notion of prevalence see Appendix A. We show that a detective with the ergodic sum method "almost certainly" gives the correct symmetry of an SBR attractor.

Theorem 3.1 *Let $\phi \in C^1_\Gamma(S,W)$ with S isotropy equivalent to M and W a distinguishing representation. Suppose that for each isotropy subgroup $G \subset G(M)$ of M and all neighbourhoods $N \in Fix^S(G)$ we have*

$$Span\{D_x\phi v : x \in N, v \in T_xN\} = Fix^W(G).$$

Then for all SBR attractors (A,ρ) there is a prevalent set of observations $\psi \in C^k_\Gamma(M,S)$ such that

$$\Sigma(K^E_{\phi,\psi}(A)) = \Sigma(A).$$

Remark 3.2 *Thus detectives for the ergodic sum method give the correct symmetry of an SBR attractor not just generically but also "almost certainly".*

We will make use of the measure transversality theorem [17, Lemma 3, page 230] which, adapted to our setting, can be stated in the following way.

Lemma 3.3 *Let $B \subset \mathbf{R}^t$ be an open set. Let $F : B \to \mathbf{R}^m$ be continuously differentiable and assume that the derivative DF has full rank at every point of B (that is to say DF is onto). Suppose Z is a subspace of codimension one or greater in $\mathbf{R}^m$. Then for almost every p in B, $F(p) \in Z^c$.*

Proof

We will identify the space $P^q_\Gamma(M,S)$ of Γ equivariant polynomials of degree at most q with $\mathbf{R}^t$ for some t and consider, for a given (A,ρ), the map

$$F(g) = \int_A \phi(g(x))d\rho(x)$$

from $P^q_\Gamma(M,S)$ to W. Note that F is continuously differentiable. Furthermore for a generic set of $g \in P^q_\Gamma(M,S)$ the differential has full rank if q is sufficiently large [15, Theorem 1.2]. Choose such a q and g' i.e. so that DF has full rank at g' and note that since this is an open condition so there is a neighbourhood B of g' such that DF restricted to this neighbourhood is onto. Hence by the Measure Transversality Lemma above for almost every $g \in B$, $F(g)$ has symmetry group equal to $\Sigma(A)$. The fact that F is linear and prevalence is translation invariant implies that almost every $g \in P^q_\Gamma(M,S)$ has the property that $F(g)$ has symmetry group equal to $\Sigma(A)$. Fix an integer $k > 0$ and let G_k denote the subset of $C^k_\Gamma(M,S)$ defined by $G_k = \{g \in C^k_\Gamma(M,S) : F(g)$ has symmetry equal to $\Sigma(A)\ \}$. Finally note that for each $k > 0$ the subspace $P^q_\Gamma(M,S) \subset C^k_\Gamma(M,S)$ serves as a probe for G_k. In fact given $p \in G_k$, we may define a one-dimensional probe by $P = \{tp : t \in \mathbf{R}\}$. It is easy to see that for each $g \in G^c_k$ the set of $t \in \mathbf{R}$ such that $g + tp \in G^c_k$ has Lebesgue measure zero.

Hence for each integer $k > 0$

$$\Sigma(K^E_{\phi,\psi}(A)) = \Sigma(A)$$

for a prevalent set of observations $\psi \in C^k_\Gamma(M,S)$.

□

Remark 3.4 *The same proof with minor modifications shows that the detectives of [5, 10, 15] "almost certainly" give the correct symmetry for both the ergodic sum and Lebesgue integral method. In fact in [5, Definition 5.1], [10, Definition 2.2] and [15, Definition 1.1] we may replace the topological notion 'open, dense' in the definition of detective by prevalence and the corresponding theorems [5, Theorem 5.2], [10, Theorem 3.3] and [15, Theorem 1.3] still are valid.*

Remark 3.5 *We have a somewhat weaker result in the case of the Lebesgue integral method as applied in this paper. In the proof of Theorem 2.8 we may show that for a prevalent set of* $g \in C^k_\Gamma(M, S)$ *the symmetry of* $\mathcal{L}(g)$ *is equal to the symmetry of* $\Sigma(A)$.

4 Detectives for parametrised families of attractors

The aim of a detective is for a given attractor A to obtain the correct answer for a generic set of equivariant observations. However, it may fail at isolated points in a persistent way if we examine perametrised families of attractors.

For systems parametrised by some vector $\lambda \in D$ open in $\mathbf{R}^d$ we want detectives that will work for parametrised families of attractors. This leads to the following definition for SBR detectives; similarly one can define for integrated observed detectives.

Definition 4.1 *A* d-parameter SBR detective *is a function* $\phi : S \to W$ *such that given any continuous family of attractors* $A(\lambda)$ *parametrised by* $\lambda \in \mathbf{R}^d$ *there exists an open dense set of observations* $\psi \in C^k_\Gamma(M, S)$ *such that for all* λ, *if* $A(\lambda)$ *is an SBR attractor then*

$$\Sigma(K^E_{\phi,\psi}(A(\lambda))) = \Sigma(A(\lambda)).$$

Note that the definition in Section 2.1 corresponds to case $d = 0$. For larger d we need to consider larger dimensional observations and distinguishing representations.

Definition 4.2 *A representation* W *of the finite group* Γ *is* d-nested *if* d *is the minimum integer such that for all isotropy subgroups* $G < H$,

$$\dim Fix(G) > \dim Fix(H) + d.$$

This means that any fixed point spaces of higher isotropy contained in $\mathrm{Fix}(G)$ are of codimension strictly greater than d.

Lemma 4.3 *A necessary condition that* $\phi : S \to W$ *is a* d*-parameter detective is that the isotropy faithful observation space* S *and the distinguishing representation* W *are* d*-nested.*

Proof Under the assumptions of continuity of $A(\lambda)$ in λ we note that $K^E_{\phi\circ\psi}(A(\lambda))$ is continuous in λ. If W is not d-nested we know that there are $G < H$ subgroups with $\mathrm{Fix}_W(H)$ of codimension less than or equal to d in $\mathrm{Fix}_W(G)$. Thus we can construct a family of attractors (all with $\Sigma(A) = G$, $T(A) = 1$) such that the image of D under K intersects $\mathrm{Fix}_W(H)$ transversely. There will then be an open set of ψ which preserve this intersection implying that all observations in this set will have at least one point of λ with the isotropy of $K(\lambda)$ equal to H.

Similarly if S is not d-nested we assume that for two isotropy subgroups $G < H$ we have $\mathrm{Fix}_W(H)$ of codimension less than or equal to d in $\mathrm{Fix}_W(G)$. We construct a family of fixed point attractors $A(\lambda) = \{x(\lambda)\}$ with $T(A) = \Sigma(A) = G$ and a ψ such that $\psi(x(\lambda))$ intersects $\mathrm{Fix}_S(H)$ transversely. This intersection will be persistent under deformation of ψ implying that there is at least one point where $T(\psi(A(\lambda))) = H$.

□

In part II we observe an example of isolated points in parameter space that are assigned the incorrect symmetry due to S not being d-nested. One can of course use detectives in the sense of Definition 2.5 for parametrised systems, but one must be aware that they can give incorrect answers on a subset of parameter space. The following characterisation of these incorrect answers can be proved as for the above lemma.

Lemma 4.4 *If we use a d-parameter detective ϕ (which is* not *a $d+1$ parameter detective) for a p parameter continuous family of attractors $A(\lambda)$ ($p > d$), then there will be at best an open dense set of observations ψ such that the detectives give the correct symmetries for a set of parameters whose complement is a set of codimension $p - d$.*

We finish this section with two examples of how detectives in the standard sense do not give the correct answers for one parameter families of attractors.

Example: Incorrect Instantaneous symmetry Consider $M = \mathbf{R}$ with $\mathbf{Z}_2$ acting by $x \mapsto -x$. If a dynamical system on M has a path of fixed points $x(\lambda)$ with trivial isotropy then for $S = \mathbf{R}$ (which is zero nested) and an equivariant observation $\psi(x) = \sin x$ we cannot avoid giving the 'wrong answer' at points where $\sin x = 0$. However by taking $S = \mathbf{Z}^2$ (two copies of the same representation) and e.g. $\psi(x) = (\sin x, \sin \sqrt{2}x)$ we avoid hitting points of higher isotropy in a one parameter family.

Example: Incorrect Average symmetry Consider again $\mathbf{Z}_2$ acting by $x \mapsto -x$, this time in $S = W = \mathbf{R}$. Suppose we have a dynamical system on some isotropy equivalent M with a continuously parametrised family of ergodic invariant measures that project onto $\mu(\lambda)$ measures supported on S, all of which are asymmetric and a detective $\phi(x) = x$. Then it is a codimension one phenomenon that the mean $\int x \, d\mu(x)$ can pass through zero on varying λ whereas it is codimension two if one considers $W = \mathbf{R}^2$ and $\phi(x) = (x, x^3)$.

5 Symmetries from Poincaré sections

We now discuss how one can relate the symmetries of attractors for flows in a phase space M to those of the intersection of the attractor with an invariant Poincaré section in M. Suppose we have a flow defined on a phase space M with a continuous evolution operator

$$\Phi(x,t) : M \times \mathbf{R}^+ \to M$$

equivariant under an action of Γ on M.

Definition 5.1 *An* invariant Poincaré section *for the attractor $A \in \mathcal{A}(M)$ is a subset P of M such that (a) $\Sigma(P) = \Gamma$ and (b) $A = \Phi(A \cap P, \mathbf{R}^+)$, i.e. A is precisely the forward evolution of its intersection with P.*

Given such a section, we relate the symmetry of the intersection of an attractor with the Poincare section to that of the attractor in the following lemma. The same symmetry group is obtained whether we measure the symmetry of the attractor A or its intersection with the Poincare section. Note that $B = A \cap P$ will be an attractor for the return map on the invariant Poincaré section P.

Lemma 5.2 *Suppose A is an attractor for Φ and $B = A \cap P$ is its intersection with P. Then $\Sigma(A) = \Sigma(B)$.*

Proof If $\gamma \in \Sigma(A)$ then $\gamma A = A$; also $\gamma P = P$ by definition of invariance and so $\gamma(A \cap P) = A \cap P$ and $\gamma \in \Sigma(B)$. Conversely, if $\gamma \in \Sigma(B)$ then $\gamma(A \cap P) = A \cap P$ and so $\gamma A = \Phi(\gamma(A \cap P), \mathbf{R}^+) = \Phi((A \cap P), \mathbf{R}^+) = A$.
$\square$

Remark 5.3 *We do not require that A possess a dense orbit or that the intersections are transverse.*

Typically for automonous systems there is no 'global Poincaré section', i.e. no section that works for all attractors. However for periodically forced systems there can be because all orbits pass transversely through sections of constant forcing angle. We use this property for the example in Part II of this paper.

6 Discussion

In summary, we have made rigorous the suggestion in [5, section 10] that to apply detectives to experiments one must take an equivariant observation and then compute the average in this space. Vital for this is Proposition 2.4 which shows that a *generic* observation in a large enough space will permit measurement of $\Sigma(A)$ and $T(A)$.

We have show that using an observation space S that is isotropy equivalent to M we can still generically detect the symmetries. If the action of the group on M is either unknown or distinguishing for Γ we may take a distinguishing representation as S, but in general it can be much smaller. For example, in a network of n coupled cells where symmetries act by permutation of the cells one may easily see that $\mathbf{R}^n$ is isotropy equivalent; thus it is necessary only to take one measurement from each cell and not to distinguish all subgroups (for example, Z_4 cannot be the isotropy of a point for the $\mathbf{S}^4$ example in Part II).

One of the difficulties associated with the implementation of detectives has been the memory requirements that the Lebesgue integral method places upon computers. To overcome this the ergodic sum method was developed, but to be made rigorous this method requires stronger assumptions on the attractor dynamics (namely the existence of an SBR measure). In addition, a difficulty associated with the ergodic sum method is the often slow convergence of the ergodic sum.

The method of averaging over the discretised observed attractor in an observation space has the advantage of fast convergence coupled with relatively small dimension of domain over which one need integrate, and this gives us an discretisation method that will even work for partial differential evolution equations with symmetry or experiments that have unknown phase spaces.

Attractors for partial differential equations In the case of partial differential evolution equations that have finite-dimensional attractors, our results still apply. Notably, if M is a Banach space and attractors are contained in an attracting invariant submanifold $M^0 \subset M$ the method of Proposition 2.4 easily gives a generic set of observables in the supremum norm topology on $C^0_\Gamma(M,S)$.

Open problems There remain many outstanding questions, for example how to characterise sufficient conditions for d-parameter detectives, and how to proceed with attractors that are not in $\mathcal{B}(A)$.

Several points concerning the notion of detective are still in some sense unsatisfactory. For example, perhaps a more natural definition of detective than the one we have given would be "ϕ is an *integrated observed detective* if for all compact A there is an open dense set B of observations $\psi \in C^k_\Gamma(M,S)$ and a positive function $\epsilon_0(\psi)$ such that

$$\Sigma(K^I_\phi(\psi(A)^\epsilon)) = \Sigma(A)$$

for all $\psi \in B$ and all $0 < \epsilon < \epsilon_0$." However it is in general impossible to obtain this uniformly for ϵ once we have fixed an observable ψ. The following example shows that as $\epsilon \to 0$ there may be countably infinite values of ϵ for which $\Sigma(K^I_\phi(\psi(A)^\epsilon))$ does not give the correct answer.

Example: A set B such that $\Sigma(K^I_\phi(\psi(B)^\epsilon)) \neq \Sigma(B)$ for a sequence of $\epsilon_i \to 0$ Let $\mathbf{Z}_2$ act on $\mathbf{R}$. Note that $\phi(x) = x$ satisfies the conditions for a detective. Choose a point x_1 and for a fixed ϵ_1 construct intervals $(x_1-\epsilon_1, x_1+\epsilon_1)$ and $(-x_1-\epsilon_1, -x_1+\epsilon_1)$. In the interval $(x_1-\epsilon_1, x_1+\epsilon_1)$ place another point x_2.

Let B_1 be the set $\{\pm x_1, x_2\}$. Note that the integral of ϕ over a δ neighbourhood of S_1 will be $\mathbf{Z}_2$ symmetric (i.e. equal zero) if $\delta > \epsilon_1$ but will be positive if $\delta < \frac{\epsilon_1}{2}$. Suppose that the value of the integral of ϕ over a δ neighbourhood of B_1 has positive value α_1 if $\delta = \epsilon_2$. Now take $x_3 \in (x_1 - \epsilon_1, x_1 + \epsilon_1)^c$ and $\epsilon_2 < \frac{\epsilon_1}{2}$ small enough so that the integral of $\phi(x) = x$ over an interval of length ϵ_2 centred on x_3 is less than α_1. Construct intervals $(x_3 - \epsilon_2, x_3 + \epsilon_2)$ and $(-x_3 - \epsilon_2, -x_3 + \epsilon_2)$. Note that if n is large enough and the points $\{-x_i\}$, $i = 3, .., n$ lie in the interval $(-x_3 - \epsilon_2, -x_3 + \epsilon_2)$ then define $S_2 = \{\pm x_3, -x_4, ..., x_n\}$. The set $B_1 \cup B_2$ is such that the integral of ϕ over a δ neighbourhood of $B_1 \cup B_2$ is zero if $\delta > \epsilon_1$, positive if $\delta = \frac{\epsilon_1}{2}$ and negative if δ is sufficiently small. We may then take a point in the complement of an ϵ_1 neighbourhood of B_1 and an ϵ_2 neighbourhood of B_2 and repeat the construction to obtain a set $B = \overline{\cup_i B_i}$ such that as $\delta \to 0$, the integral of ϕ over a δ neighbourhood of B oscillates infinitely often between positive and negative values.

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A Appendix: prevalence

The key idea behind the notion of prevalence derives from the following observation, [17, page 219]: Let $S \subset \mathbf{R}^n$ be a Borel set. If there exists a probability measure μ with compact support such that every translate of S has μ-measure zero, then S has Lebesgue measure zero. Suppose that V is a complete metric linear space. We write $S + v$ for the translate of a set $S \subset V$ by a vector $v \in V$. The generalisation of this idea to the infinite-dimensional setting leads naturally to the following definition.

Definition A.1 *A measure μ is said to be transverse to a Borel set $S \subset V$ if the following two conditions hold:*

1) There exists a compact set $U \subset V$ for which $0 < \mu(U) < \infty$

2) $\mu(S + v) = 0$ for every $v \in V$.

In $\mathbf{R}^n$ those subsets S which have a measure transverse to them are precisely the subsets of zero n-dimensional Lebesgue measure. In an infinite-dimensional setting subsets having a measure transverse to them play the analogous role to subsets of zero Lebesgue measure. These sets are called shy sets- the complement of a shy set is called a prevalent set.

Definition A.2 *A Borel set $S \subset V$ is called shy if there exists a measure transverse to S. The complement of a shy set is called a prevalent set.*

A useful way to show that a set is prevalent is to use what is termed a probe.

Definition A.3 *We call a finite-dimensional subspace $P \subset V$ a probe for a set $T \subset V$ if Lebesgue measure supported on P is transverse to a Borel set which contains the complement of T.*

Thus a sufficient condition for T to be prevalent is that T has a probe.

B Appendix: proof of theorem 2.8

For the SBR method we give a constructive proof, i.e. without using a implicit function theorem as in [5, 10, 15].

Define the map $\Phi_A : C^k(M, S) \to W$ by

$$\Phi_A(\psi) := \int_A \phi \circ \psi d\rho.$$

where (A, ρ) is an SBR attractor in M and a projection $P^W_\Sigma : W \to W$ by

$$P^W_\Sigma(w) = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \sigma w.$$

Proof (of Theorem 2.8: **Ergodic sum method**) We assume that (A, ρ) is an SBR attractor in M (recall that $A \in \mathcal{A}(M)$). Suppose that $G = T(A)$ and $\Sigma(A) = \Sigma$. We define the subset of observations in $C^k_\Gamma(M, S)$ that give the correct symmetry in the following way:

$$\begin{aligned} \mathcal{P} &:= \{\psi \ : \ \Sigma(K^E_{\phi,\psi}(A)) = \Sigma\} \\ &= \{\psi \ : \ d(\Phi_A(\psi), \rho\Phi_A(\psi)) = 0 \text{ if and only if } \rho \in \Sigma\}. \end{aligned}$$

From the second line, by continuity of Φ_A with $\psi \in C^k_\Gamma(M, S)$ it is apparent that $\mathcal{P}$ is open.

It remains to prove that $\mathcal{P}$ is dense. Choose any $\psi \in C^k_\Gamma(M, S)$ and any $a \in A$ with isotropy G. Note that if W_G is the stratum (set of all points) with isotropy G in W then for all $y \in W_G$ the isotropy of $P^W_\Sigma(y)$ is precisely Σ.

Because of condition (2.1) it is clear that the span of the derivative $D\phi_x$ intersects W_G for a dense set of points $x \in S$. Thus it is possible to find ψ_0 arbitrarily close to ψ such that the span of $D\phi_{\psi_0(a)}$ intersects W_G. Choose any v such that $D\phi_{\psi_0(a)}(v)$ is in W_G. By continuity of $D\phi$, for all small enough δ there exists an open cone $\mathcal{C}$ in W_G such that

$$\{D\phi_{\psi(x)} v \ : \ x \in B_\delta(a)\} \subset \mathcal{C}.$$

Also for small enough δ the fact that $A \in \mathcal{A}(M)$ means that Σ/G-orbit of $B_\delta(a)$ consists of disjoint balls.

We define

$$\psi_\epsilon(x) = \psi_0(x) + \epsilon \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} (\gamma v)\eta(\gamma^{-1}(x))$$

where $\eta \in C^k(M, \mathbf{R})$ is a non-negative function with $\eta(a) = 1$ and $\eta(x) = 0$ for $x \in B_\delta(a)^c$.

Observe that for given δ and $\eta(x)$

$$\begin{aligned}
\Phi_A(\psi_\epsilon) &= \Phi_A(\psi_0) + \epsilon \int_A D\phi_{\psi_0(x)} \left(\frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} (\gamma v)\eta(\gamma^{-1}(x)) \right) dx + O(\epsilon^2) \\
&= \Phi_A(\psi_0) + \epsilon P_\Sigma^W \int_{x\in B_\delta(a)} (D\phi_{\psi_0(x)} v)\eta(x)\, dx + O(\epsilon^2) \\
&= \Phi_A(\psi_0) + \epsilon P_\Sigma^W y + O(\epsilon^2)
\end{aligned}$$

where $y \in \mathcal{C} \subset W_G$. We have used the fact that η is non-zero only in a neighbourhood of $B_\delta(a)$ and that the integral of a set of vectors in the cone $\mathcal{C}$ will also be contained in that cone. Writing $\Phi_A(\psi_0) = P_\Sigma^W z$ for some z, for small enough ϵ, $\Phi_A(\psi_\epsilon)$ has isotropy Σ. Noting that $\|\psi_0 - \psi_e\| < \epsilon\|\eta\|$ in the C^k norms completes the proof of density of $\mathcal{P}$.

□

Proof (of Theorem 2.8: **Integrated observed method**)

Suppose $C \subset S$ with $T(C) = G$. Let m denote Lebesgue measure on $\mathrm{Fix}_W(G)$ and suppose C has positive m measure.

If $\sigma C = C$ then $\sigma \int_C \phi d\ell = \int_C \phi d\ell$ and thus $\int_C \phi d\ell$ lies in $\mathrm{Fix}_W(\Sigma(C))$. Clearly the condition that $\int_B \phi d\ell$ does not lie in a fixed point subspace of some group $\Delta \supset \Sigma(A)$ is an open condition on the space of sets B with respect to the d_m topology. Thus we need only show that it is dense. To this end define the map $\mathcal{L}$ from the space of Γ-equivariant diffeomorphisms of S with the C^k topology $\mathrm{Diff}^k_\Gamma(S)$ to W by

$$\mathcal{L}(g) = \int_C \phi \circ g |\mathrm{Jac}(g)| d\ell$$

where $g \in \mathrm{Diff}^k_\Gamma(S)$. Note that by a change of variables $\int_C \phi\circ g|\mathrm{Jac}(g)|d\ell$ is equal to $\int_{gC} \phi d\ell$. Furthermore the linearisation of $\mathcal{L}(g)$ is the same as the linearisation of the map $g \to \int_{\psi(A)^\epsilon} \phi\circ g d\ell$ and the same linearisation argument as that used in [5, 10, 15] shows that this map is generically onto so there exists a near-identity diffeomorphism of S, call it g', so that the isotropy subgroup of $\int_{g'C} \phi d\ell$ precisely equals $\Sigma(C)$. Thus the condition is also dense since if g' is a near-identity diffeomorphism then $g'C$ is close to C in the d_m topology.

□

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