LIVŠIC REGULARITY FOR MARKOV SYSTEMS

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Abstract. We prove measurable Livšic theorems for dynamical systems modelled by Markov towers. Our regularity results apply to solutions of cohomological equations posed on Hénon-like mappings and a wide variety of nonuniformly hyperbolic systems. We consider both Hölder cocycles and cocycles with singularities of prescribed order.

1. Introduction

In this paper we study the regularity of solutions ψ of the cohomological equation

$$\varphi = (\psi \circ T)\psi^{-1} \qquad \mu\text{-a.e.} \tag{1}$$

where (T, X, μ) is a dynamical system and $\varphi : X \to G$ is a cocycle taking values in a Lie group G. Measurable rigidity in this context means that a measurable solution ψ must have a higher degree of regularity, in many contexts inheriting the regularity of φ and/or T. Such cohomological equations come up in different applications: they are used to determine whether certain observables have positive variance in the context of the Central Limit Theorem and related distributional theorems. In the context of group extensions, they decide on (stable) ergodicity and weak-mixing of the system. In other contexts, cohomological equations play a role in the question of whether two dynamical systems are (Hölder or smoothly) conjugate to each other.

Fundamental work on the regularity of measurable solutions to cohomological equations was done by Livšic [27, 28] who established rigidity theorems for Abelian group extensions of Anosov systems with an absolutely continuous invariant measure. Dynamical rigidity theorems are often called Livšic theorems in the literature because of this. Parry & Pollicott [37], using a

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transfer operator approach, extended Livšic's results to prove Hölder regularity of coboundary and transfer functions for compact Lie group extensions of subshifts of finite type and hence, via Markov partitions, Axiom A systems. Further generalizations for uniformly hyperbolic smooth systems are given in [31, 32, 38, 42, 43]. In the Anosov setting de la Llave et al [14] prove a C^∞ version of Livšic's theorem and also C^∞ dependence of solutions upon parameters.

There are only a few results on Livšic theorems for nonuniformly hyperbolic or discontinuous systems. Pollicott & Yuri [39] have established Livšic theorems for Hölder $\mathbb{R}$-extensions of β-transformations ($f : [0,1) \longrightarrow [0,1)$, $f(x) = \beta x \pmod 1$ where $\beta > 1$) via transfer operator techniques but the regularity they obtain is bounded variation rather than Hölder. Jenkinson [20] has proved that essentially bounded measurable coboundaries ψ (i.e. solutions to $\varphi = \psi \circ f - \psi$) for $\mathbb{R}$-valued smooth cocycles φ over smooth expanding maps f have smooth versions.

Nicol & Scott [33] have obtained Livšic theorems for certain discontinuous hyperbolic systems, showing that coboundary solutions taking values in Lie groups satisfying a pinching condition (to ensure that the system is partially hyperbolic) are Lipschitz if the cocycle is Lipschitz. The same techniques show that, for such systems, measurable transfer functions taking values in compact matrix groups have Lipschitz versions. These results were applied to prove stable ergodicity for semisimple and Abelian compact group extensions of certain uniformly hyperbolic systems with singularities, including the β-transformation, Markov maps and mixing Lasota-Yorke maps.

Aaronson & Denker [1, Corollary 2.3] have shown that if $(f, X, \mu, \mathcal{P})$ is a mixing Gibbs-Markov map preserving a probability measure μ with countable Markov partition $\mathcal{P}$ and $\varphi : X \to \mathbb{R}^d$ is Lipschitz (with respect to a metric d on X derived from the symbolic dynamics) then any measurable solution $\varphi : X \to \mathbb{R}^d$ to $\varphi = \psi \circ f - \psi$ has a version $\tilde{\varphi}$ which is Lipschitz continuous, i.e. there exists $C > 0$ such that $d(\tilde{\varphi}(x), \tilde{\varphi}(y)) \leq C\rho(x, y)$ for all $x, y \in f(a)$ and each $a \in \mathcal{P}$.

The work of [2] is a study of the statistical properties of fibred systems and gives rigidity results which provide checkable conditions for the aperiodicity of cocycles (i.e. nonexistence of solutions ψ) which allow one to establish, for example, de Moivre's approximation for various systems, including the β-transformation. A related result is given in [16, Lemma 6.1.2].

In two influential papers [45, 46], Young describes properties of a class of Markov extensions (which we will call Young towers) which are useful to establish rates of decay of correlations and the CLT in non-uniformly hyperbolic systems. Scott [40, 41] has recently proved measurable Livšic theorems for certain Lie group valued Hölder cocycles over a class of unimodal maps modelled by a Young tower [45]. More precisely suppose (f, X, μ) (X a finite collection of intervals) is a unimodal map (belonging to a certain class) and $g : X \to G$ is a Lie group valued Hölder cocycle (satisfying a pinching condition if G is noncompact). If ψ is a measurable solution to $g = (\psi \circ f)\psi^{-1}$ μ-a.e. then ψ is Hölder on an arbitrarily large open set (i.e. given $\varepsilon > 0$ there exists an open set U such that ψ is Hölder on U and $\mu(U) > 1 - \varepsilon$). Similar measurable Livšic theorems for other types of cohomological equations are given.

This paper extends the results of Scott in several directions. We prove measurable Livšic theorems for more general Markov extensions and for cocycles with singularities of prescribed order. We also obtain regularity results for measurable conjugacies between certain non-uniformly hyperbolic systems and measurable Livšic theorems for certain non-uniformly hyperbolic systems (in particular Hénon-like mappings).

After writing this paper we learnt that Gouëzel [18] has obtained similar results for cocycles into Abelian groups over one-dimensional Gibbs-Markov systems and Young towers. The cocycles he considers are locally Lipschitz with respect to a tower metric which in many cases is enough for applications. Our regularity results are in terms of the smooth Riemannian metric on the underlying dynamical system rather than a symbolic metric. From [18] we learnt of a Martingale Density Theorem (see Appendix) which allows a more elegant approach in part of our proof than the argument using Lebesgue density points adapted from [33].

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2. Cohomological equations and group extensions

In this section we collect some facts about group extensions and cohomological equations that set the framework of this paper. All Lie groups are assumed connected and finite-dimensional.

Suppose $f : X \to X$ is an ergodic dynamical system with respect to an invariant measure μ. Let G be a compact Lie group and let dh denote Haar measure on G. Suppose $\varphi : X \to G$ is measurable.

Proposition 1. [25, 34]

(1) *The compact group extension $T(x, g) = (f(x), \varphi(x)g)$ is ergodic with respect to $\mu \times dh$ if and only if the equation*

$$\psi(fx) = R(\varphi(x))\psi(x) \qquad \mu\text{-}a.e. \tag{2}$$

where R is an irreducible (unitary) representation of degree d and $\psi : X \longrightarrow \mathbb{C}^d$ is measurable, is only satisfied when ψ is constant or R is the trivial representation.

(2) *Suppose $f : X \to X$ is weak-mixing and $T : X \times G \to X \times G$ is ergodic. Then $T(x, g) = (f(x), \varphi(x)g)$ is weak-mixing with respect to $\mu \times dh$ if and only if for any $e^{i\alpha} \neq 1$ and any non-trivial one-dimensional representation χ of G the equation*

$$\psi(fx) = e^{i\alpha}\chi(\varphi(x))\psi(x) \ \ \mu\text{-}a.e. \tag{3}$$

has no nontrivial measurable solution $\psi : X \to \mathbb{C}$.

Note that the aperiodicity condition of [2]

$$\gamma \circ \varphi = \lambda\psi/\psi \circ f$$

where γ is a character of G (a locally compact Abelian polish group), $\lambda \in S^1$ is a special case of equation (3).

Suppose that $\varphi_i : X \to G$, $i = 1, 2$ are two compact Lie-group valued cocycles over a system (f, X, μ). A measurable function $\psi : X \to G$ conjugates the G extensions $T(x, g) = (fx, \varphi_i(x)g)$ $(i = 1, 2)$ if

$$\psi(fx)\varphi_1(x) = \varphi_2(x)\psi(x) \qquad \mu\text{-a.e.} \tag{4}$$

We call such a conjugating function ψ a *transfer function.*

If G is compact we may identify G with a subgroup of $U(d)$, the group of $d \times d$ unitary matrices. In this representation we may identify a G-valued

$\varphi : X \to G$ with $\varphi : X \to \mathbb{C}^{d^2}$. Define $\theta(x) : M(d) \to M(d)$, a mapping from the space of $d \times d$ complex matrices to itself by

$$\theta(x) : A \to \varphi_2(x) A \varphi_1(x)^*.$$

It is possible to show that $\theta(x) : M(d) \to M(d)$ is unitary. There is a standard way, see [37, Theorem 1] and [20, Theorem A] to rewrite $\psi(fx)\varphi_1(x) = \varphi_2(x)\psi(x)$ in form $\psi(fx) = \theta(x)\psi(x)$ where $\psi : X \to \mathbb{C}^{d^2}$, $\theta(x) \in U(d)$. Hence the question of the regularity of conjugacies between compact group extensions may be reduced to those of the regularity of solutions to equation (2).

The proof of our coboundary Livšic regularity results, such as Theorem 1, may be slightly modified (as in [33, Section 2.1]), to establish the same degree of regularity for solutions ψ to equation (2), equation (3) or equation (4) posed over the same dynamical system. We omit the straightforward proof of this and refer the reader to [33, Section 2.1].

2.1. **Lie groups.** Let G be a connected Lie group with Lie algebra denoted by $L(G)$ which we identify with the tangent space at the identity, T_eG. We let r_g denote right multiplication by $g \in G$. Given a norm $\|\cdot\|$ on T_eG we define a norm on T_gG by $\|\mathbf{v}\|_g = \|r_{g^{-1}}\mathbf{v}\|_e$. This norm induces a right invariant metric d_G on G so that $d_G(gk, hk) = d_G(g, h)$, see [38, Section 4]. Throughout this paper we will write $d(\cdot,\cdot)$ instead of $d_G(\cdot,\cdot)$ when it is clear from context that we mean the metric on G. For a general reference on Lie groups see [7].

We define the adjoint map $\mathrm{Ad}\colon G \to \mathrm{Aut}(L(G))$, for $g \in G$ and $X \in T_eG$ by

$$\mathrm{Ad}(g)\mathbf{v} = \frac{d}{dt}(g \exp(t\mathbf{v}) g^{-1}) \text{ at } t = 0.$$

Note that when G is a matrix group this action is conjugation i.e. $v \to gvg^{-1}$. A calculation [38, Section 4] shows that

$$d(gh, gk) \leq \|\mathrm{Ad}(g)\| d(h, k). \tag{5}$$

Suppose $\varphi\colon M \to G$ is Hölder of exponent $\alpha > 0$. Define

$$\mu_u := \lim_{n\to\infty} \left(\sup_{x \in X} \|\mathrm{Ad}(\varphi_n(x))\| \right)^{\frac{1}{n}},$$

where $\varphi_n(x) = \varphi(f^{n-1}x) \ldots \varphi(fx)\varphi(x)$.

If G is Abelian or compact, then $\sup_x \| \operatorname{Ad}(\varphi_n(x))\|$ is bounded in n, whereas if G is nilpotent, $\sup_x \| \operatorname{Ad}(\varphi_n(x))\|$ can grow at most at a polynomial rate. In these three cases, $\mu_u = 1$.

3. Axiomatic Approach for Nonuniformly Expanding Maps

Let (M, ρ, μ) be a metric space endowed with a non-atomic Borel probability measure μ. We assume M can be decomposed as $M = \cup_k M_k \bmod \mu$, where each M_k is connected and $\sup_k(\operatorname{diam}(M_k)) \leq 1$. Let $f : \cup_k M_k \to M$ be a map such that $f|M_k$ is continuous for each k and such that μ is f-invariant and ergodic.

Let $\mathcal{P}_0$ be the partition of M into the sets M_k, and $\mathcal{P}_n = \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P}_0)$. For $x \in M$, let $\mathcal{P}_n[x]$ be the partition element (cylinder set) in $\mathcal{P}_n$ containing x.

Consider the natural extension $(\hat{M}, \hat{f}, \hat{\mu})$ of (M, f, μ): each point $\hat{x} \in \hat{M}$ is a sequence

$$\hat{x} = (x_0, x_1, x_2, \dots) \text{ with } M \ni x_i = f(x_{i+1}) \text{ for all } i \geq 0.$$

The measure $\hat{\mu}$ is defined in the standard way [22] and in particular for each n:

$$\hat{\mu}(\{\hat{x} \in \hat{M} \mid x_n \in A\}) = \mu(A)$$

for each μ-measurable set A.

We assume:

(1) For all k, $f : M_k \to f(M_k)$ is one-to-one and $f(M_k)$ is equal to a union of components $M_l \bmod \mu$ (Markov property).

(2) There exists $\lambda > 1$ and for $\hat{\mu}$-a.e. $\hat{x}$ a number $K(\hat{x})$ such that

$$\rho(y_n, z_n) \leq K(\hat{x})\lambda^{-n}\rho(y_0, z_0) \tag{6}$$

for all $n \geq 0$ and $y_n, z_n \in \mathcal{P}_n[x_n]$. (In dimension one this assumption can be weakened, see Section (4)).

(3) Let $J_\mu(x)$ denote the Jacobian of μ at x. For $\hat{x} \in \hat{M}$, define $J_\mu^n(x_n) = \prod_{i=1}^n J_\mu(x_i)$. For $\hat{\mu}$-a.e. $\hat{x}$, there exists a constant $C(\hat{x})$ such that if $\hat{y}, \hat{z} \in \hat{M}$ are such that $y_i, z_i \in \mathcal{P}_i[x_i]$ for all i, then

$$\left| \frac{J_\mu^n(y_n)}{J_\mu^n(z_n)} \right| \leq C(\hat{x}). \tag{7}$$

The Jacobian J_μ of μ is γ-Hölder if there exists C and $\gamma \in (0,1]$ such that

$$\left|\frac{J_\mu(x)}{J_\mu(y)} - 1\right| \leq C \cdot \rho(f(x), f(y))^\gamma. \tag{8}$$

Hölderness of the Jacobian implies a result stronger than (7).

Lemma 1. *Assume J_μ is γ-Hölder. For $\hat{\mu}$-a.e. $\hat{x}$, there exists a constant $B = B(\hat{x}, C, \lambda^\gamma)$ such that*

$$\left|\frac{J_\mu^n(y_n)}{J_\mu^n(z_n)}\right| \leq 1 + B(\hat{x}, \lambda^\gamma)\rho(y_0, z_0)^\gamma.$$

Proof. Using (8) and (6), we obtain

$$\begin{aligned}
\left|\frac{J_\mu^n(y)}{J_\mu^n(z)}\right| &= \prod_{i=0}^{n-1} \frac{|J_\mu(y_i)|}{|J_\mu(z_i)|} \\
&\leq \prod_{i=0}^{n-1} (1 + C\rho(f(y_i), f(z_i))^\gamma) \\
&\leq \exp(C \sum_{i=1}^{n} \rho(y_i, z_i)^\gamma) \\
&\leq \exp(C \cdot K(\hat{x}) \sum_{i=1}^{n} \lambda^{-i\gamma} \rho(y_0, z_0)^\gamma) \\
&\leq \exp\left(\frac{C \cdot K(\hat{x})}{\lambda^\gamma - 1} \rho(y_0, z_0)^\gamma\right),
\end{aligned}$$

which is smaller than $1 + B\rho(y_0, z_0)^\gamma$ for some B depending only on $\hat{x}$, C, λ^γ and the diameter of the component M_k containing x_0.

In fact, the above computation only requires that $\sum_{i=0}^{n-1} \operatorname{diam}(\mathcal{P}_i[x_i])^\gamma \leq K_0(\hat{x}) < \infty$ for μ-a.e $\hat{x}$ independently of n, which is an estimate valid under a less strict assumption than (6). □

Cocycle assumptions: Let $\varphi\colon M \to G$ be Hölder of exponent $\alpha > 0$. Recall that

$$\mu_u := \lim_{n\to\infty} \left(\sup_{x \in M} \| \operatorname{Ad}(\varphi_n(x))\|\right)^{\frac{1}{n}},$$

where $\varphi_n(x) = \varphi(f^{n-1}x) \ldots \varphi(fx)\varphi(x)$. If G is Abelian $\| \operatorname{Ad}(\varphi_n(x))\| = 1$ and $\| \operatorname{Ad}(\varphi_n(x))\|$ is bounded if G is compact. For nilpotent groups G, $\| \operatorname{Ad}(\varphi_n(x))\|$ grows at most at a polynomial rate in n, so $\mu_u = 1$. For the general case, we impose a partial hyperbolicity condition (PH) on the group extension:

$$1 \leq \mu_u < \lambda^\alpha \tag{PH}$$

where λ is from (6).

Theorem 1. *Assume that (M, f, μ) is a measure preserving Markov system as above and let $M_k \in \mathcal{P}_0$. Let $\varphi : M \to G$ be a Lie group valued α-Hölder observable (i.e. $d(\varphi(x), \varphi(y)) \leq \rho(x, y)^\alpha$) satisfying the partial hyperbolicity condition (PH) above. Let $\psi : M \to G$ be a μ-measurable solution of the cohomological equation*

$$\psi \circ f(x) = \varphi(x) \cdot \psi(x) \qquad \mu\text{-a.e.}$$

Then there is a version $\tilde{\psi}$ of ψ (i.e. $\psi = \tilde{\psi}$ μ-a.e.) such that $\tilde{\psi}$ is α-Hölder on M_k.

Corollary 1. *If $f^j(\cup_{k \in S} M_k) = M$ for some $j > 0$ and finite collection of indices S then there is a version which is α-Hölder on M.*

Proof of Corollary 1. By considering the cohomological equation $\psi \circ f^n(x) = \varphi(f^{n-1}x) \cdot \varphi(f^{n-1}x) \ldots \varphi(x) \cdot \psi(x)$ we may extend the version of ψ as a Hölder function to any image $f^j(M_k)$. □

Remark 1. *It is easy to show that given $\varepsilon > 0$ there is a version of ψ which is α-Hölder on a finite union of sets $\cup_{k \in S} M_k$ such that $\mu(\cup_{k \in S} M_k) > 1 - \varepsilon$. The Hölder coefficient depends in general upon S but the exponent is uniform.*

Remark 2. *A slight modification of the proof shows that the same regularity results hold for solutions ψ to equation (2), equation (3) or equation (4).*

Proof of Theorem 1. Choose any $\Lambda := M_k \in \mathcal{P}_0$ such that $\mu(\Lambda) > 0$. Let $0 < \delta < 1$. As a consequence of the Martingale Density Theorem (see Appendix) for $\hat{\mu}$-a.e. $\hat{x} \in \hat{M}$ and for infinitely many n :

$$\frac{\mu\{y_n \in \mathcal{P}_n[x_n] : d(\psi(y_n), \psi(x_n)) < \delta\}}{\mu(\mathcal{P}_n[x_n])} > 1 - \delta.$$

Let $\hat{x}$ be such a point with $x_0 \in \Lambda$. For simplicity of notation in the rest of the proof we will not indicate the dependence of constants upon $\hat{x}$. We consider points $\hat{y} = (y_0, y_1, \ldots y_n \ldots)$ and $\hat{z} = (z_0, z_1, \ldots z_n \ldots) \in \hat{M}$ such that $y_n, z_n \in \mathcal{P}_k[x_n]$ for all $n = 0, 1, \ldots$. Hence $\hat{x}$, $\hat{y}$ and $\hat{z}$ are all paths in the "same inverse branch" of f.

On Λ we define a function $\Phi : \Lambda \to G$ by

$$\Phi(y_0) = \lim_{n\to\infty} \varphi_n(y_n)\varphi_n(x_n)^{-1},$$

where $\varphi_n(x_n) = \varphi(x_1)\ldots\varphi(x_n)$. This function is well defined since

$$\begin{aligned}
&d(\varphi_{n+1}(y_{n+1})\varphi_{n+1}(x_{n+1})^{-1}, \varphi_n(y_n)\varphi_n(x_n)^{-1}) \\
&\quad = d(\varphi_n(y_n)\varphi(y_{n+1})\varphi(x_{n+1})^{-1}\varphi_n(x_n)^{-1}, \varphi_n(y_n)\varphi_n(x_n)^{-1}) \\
&\quad = d(\varphi_n(y_n)\varphi(y_{n+1})\varphi(x_{n+1})^{-1}\varphi_n(x_n)^{-1}, \varphi_n(y_n)\varphi(x_{n+1})\varphi(x_{n+1})^{-1}\varphi_n(x_n)^{-1}) \\
&\quad = d(\varphi_n(y_n)\varphi(y_{n+1}), \varphi_n(y_n)\varphi(x_{n+1})) \qquad \text{(right invariance)} \\
&\quad \le \|\operatorname{Ad}(\varphi_n(y_n))\|\, d(\varphi(y_{n+1}), \varphi(x_{n+1})) \qquad \text{(by (5))} \\
&\quad \le C \cdot K(\hat{x}) \left((\mu_u)\lambda^{-\alpha}\right)^n \qquad \text{(by (6))} \\
&\quad \le C \cdot K(\hat{x}) \cdot \kappa^n,
\end{aligned}$$

where $\kappa \in (0, 1)$ by (PH). Thus the sequence $\varphi_n(y_n)\varphi_n(x_n)^{-1}$ is Cauchy and so converges.

Next we show that Φ is Hölder. Let $y_0, z_0 \in \Lambda$, then

$$\begin{aligned}
&d(\varphi_n(y_n)\varphi_n(x_n)^{-1}, \varphi_n(z_n)\varphi_n(x_n)^{-1}) \\
&\quad = d(\varphi_n(y_n), \varphi_n(z_n)) \qquad (9) \\
&\quad \le \sum_{i=0}^{n-1} d(\varphi_i(y_i)\varphi(y_{i+1})\varphi_{n-i-1}(z_{i+1}), \varphi_i(y_i)\varphi(z_{i+1})\varphi_{n-i-1}(z_{i+1})) \\
&\quad \le \sum_{i=0}^{n-1} \|\operatorname{Ad}(\varphi_i(y_i))\|\, d(\varphi(y_{i+1}), \varphi(z_{i+1})) \\
&\quad \le \sum_{i=0}^{n-1} C \cdot K(\hat{x})(\mu_u)^{i+1}\lambda^{-(i+1)\alpha}\rho(y_0, z_0)^\alpha.
\end{aligned}$$

Letting $n \to \infty$ gives $d(\Phi(y_0), \Phi(z_0)) \le C \cdot K(\hat{x}) \cdot \rho(y_0, z_0)^\alpha$. It is clear that if φ is Lipschitz (i.e. $\alpha = 1$) then Φ is also Lipschitz.

Define

$$\Psi_n(y_0) = \varphi_n(y_n)\varphi_n(x_n)^{-1}.$$

Then

$$\begin{aligned}\psi(y_0) &= \varphi_n(y_n)\psi(y_n)\\ &= \Psi_n(y_0)\varphi_n(x_n)\psi(x_n)\psi(x_n)^{-1}\psi(y_n)\\ &= \Psi_n(y_0)\psi(x_0)\psi(x_n)^{-1}\psi(y_n).\end{aligned}$$

Thus

$$\begin{aligned}d(\psi(y_0),\psi(z_0)) &\leq d(\Psi_n(y_0)\psi(x_0)\psi(x_n)^{-1}\psi(y_n),\Psi_n(y_0)\psi(x_0))\\ &+ d(\Psi_n(y_0)\psi(x_0)\psi(x_n)^{-1}\psi(z_n),\Psi_n(y_0)\psi(x_0))\\ &+ d(\Psi_n(y_0)\psi(x_0)\psi(x_n)^{-1}\psi(z_n),\Psi_n(z_0)\psi(x_0)\psi(x_n)^{-1}\psi(z_n)).\end{aligned}$$

By right-invariance of the metric the last term may be written as

$$d(\Psi_n(y_0),\Psi_n(z_0)).$$

As a function of y_0, $\Psi_n(y_0)$ converges to the α-Hölder function $\Phi(y_0)$. Thus letting $n\to\infty$, we obtain $d(\Psi(y_0),\Psi(z_0))\leq C\rho(y_0,z_0)^\alpha$.

Given $\eta>0$ there exists $\delta_\eta>0$ such that $d(\psi(z_n),\psi(y_n))\leq\delta_\eta$ implies

$$\begin{aligned}d(\Psi_n(y_0)\psi(x_0)\psi(x_n)^{-1}\psi(y_n),\Psi_n(y_0)\psi(x_0)) &\leq \frac{\eta}{2},\\ d(\Psi_n(y_0)\psi(x_0)\psi(x_n)^{-1}\psi(z_n),\Psi_n(y_0)\psi(x_0)) &\leq \frac{\eta}{2}.\end{aligned}$$

Choose n sufficiently large so that

$$\frac{\mu\{y_n\in\mathcal{P}_n[x_n]:d(\psi(y_n),\psi(x_n))<\delta_\eta\}}{\mu(\mathcal{P}_n[x_n])}>1-\delta_\eta.$$

Now we estimate $\mu(f^n(\{y_n\in\mathcal{P}_n[x_n]:d(\psi(y_n),\psi(x_n))<\delta_\eta\}))$ relative to $\mu(\Lambda)$. By boundedness of distortion of the Jacobian of f^n we have that

$$\begin{aligned}&\frac{\mu(f^n(\{y_n\in\mathcal{P}_n[x_n]:d(\psi(y_n),\psi(x_n))<\delta_\eta\}))}{\mu(f^n\mathcal{P}_n[x_n])}\\ &\qquad\leq\mathcal{O}(1)\;\frac{\mu(\{y_n\in\mathcal{P}_n[x_n]:d(\psi(y_n),\psi(x_n))<\delta_\eta\})}{\mu(\mathcal{P}_n[x_n])}.\end{aligned}\tag{10}$$

Hence for the above $\eta>0$, choosing δ_η smaller if necessary, we have

$$\mu(f^n(\{y_n\in\mathcal{P}_n[x_n]:d(\psi(y_n),\psi(x_n))<\delta_\eta\}))>(1-\eta)\mu(\Lambda).$$

Since $d(\psi(y_n),\psi(z_n))<\delta_\eta$ implies $d(\psi(z_0),\psi(y_0))<\eta+K\rho(z_0,y_0)^\alpha$ we have shown that

$$\begin{aligned}\mu\times\mu\{(y_0,z_0)\in\Lambda\times\Lambda:d(\psi(z_0),\psi(y_0))<2\eta+2K\rho(z_0,y_0)^\alpha\}&\\ >(1-2\eta)\mu\times\mu(\Lambda\times\Lambda).&\end{aligned}$$

Since η was arbitrary, $\psi \mid \Lambda$ has a Hölder version. $\square$

3.1. **Cocycles with singularities.** Let $\varphi : M \to G$ be a cocycle which is Hölder except for discontinuities and singularities concentrated on a finite set $\mathcal{C}$. Let λ be as in (6), which is defined μ-a.e, and for a fixed $\delta > 0$ let $B(c, \delta)$ denote a δ-neighbourhood of c for $c \in \mathcal{C}$. We consider the following three scenarios:

(1) **Bounded discontinuity:** The cocycle $\varphi(x)$ is bounded and γ-Hölder in the complement of $\mathcal{C}$ but for each $c \in \mathcal{C}$ we have: $\lim_{x\to c^+} \varphi(x) \neq \lim_{x\to c^-} \varphi(x^-)$.
(2) **Logarithmic singularity:** For each $c \in \mathcal{C}$ inside $B(c;\delta)$ we have $\|\varphi(x)\| \approx |\log \rho(x,c)|$, where $\| \ \|$ denotes the norm of the group element. Moreover there is a sequence $\{\varepsilon_n\}$, such that $\sum_n \mu(B(c;\varepsilon_n)) < \infty$, and
$$\limsup_{n\to\infty} \frac{\log \varepsilon_n^{-1}}{n \log \lambda} < 1.$$
(3) **Pole:** For each $c \in \mathcal{C}$ there is $p > 0$ so that on $B(c;\delta)$, $\|\varphi(x)\| \approx \rho(x,c)^{-p}$, and for some sequence $\{\varepsilon_n\}$, such that $\sum_n \mu(B(c;\varepsilon_n)) < \infty$, we have
$$\lim \sup_{n\to\infty} \frac{(p+1)\log \varepsilon_n^{-1}}{n \log \lambda} < 1.$$

In each case, we will assume that the Holder exponent of φ restricted to the complement of $\cup_{c\in\mathcal{C}} B(c;\delta)$ is γ. We state the following result:

Theorem 2. *Assume that (M, f, μ) is a measure preserving smooth Markov system as defined in Section 3, and $M_k \in \mathcal{P}_0$. Let $\varphi : M \to G$ be a Lie group valued observable which has a singularity set $\mathcal{C} \subset M \setminus \cup_k \overline{M_k}$, characterized by cases either (1), (2) or (3) above. Let ψ be a μ-measurable solution of the cohomological equation*
$$\psi \circ f(x) = \varphi(x)\psi(x) \qquad \mu\text{-a.e.}$$
Then there is a version $\tilde{\psi}$ of ψ, with $\tilde{\psi} = \psi$ μ-a.e such that $\tilde{\psi}$ is α-Hölder on M_k, for some $\alpha \in (0, 1)$.

Remark 3. *The condition that $\mathcal{C} \subset M \setminus \cup_k \overline{M_k}$ is unnecessarily strong, but is shared by many examples. For multimodal maps and Lorenz maps on the interval, induced maps F to a neighborhood Y of the critical point c are common constructions* [10]. *The interval Y has the decomposition*

$Y = \cup_i Y_i \bmod \mu$ *where* $F|Y_i = f^{r_i} : Y_i \to Y$ *is monotone onto for an appropriate* $r_i > 0$. *For each* i, $\overline{f^j(Y_i)} \not\ni c$ *for* $0 \le j < r_i$. *Hence, if* φ *has only singularities at c, then using the above theorem and by the argument of Corollary 1, we can conclude that* ψ *has a Hölder version on* Y.

Remark 4. *Also in cases where* $\mathcal{C} \subset M \backslash \cup_k \overline{M_k}$ *fails, the proof below can still be used to get partial results. Given an element* M_k *the proof constructs an* N *and a component* J *(where* $f^N(J) = M_k$*) of the preimage* $f^{-N}(M_k)$ *such that* ψ *has a Hölder version on* J. *If* J *can be chosen such that* $\overline{f^i(J)} \cap \mathcal{C} = \emptyset$ *for* $0 \le i < N$, *then there is a version of* ψ *which is Hölder on* M_k. *If this condition is not satisfied then using the proof of Corollary 1 it is possible to show that for any* $\varepsilon > 0$, M_k *contains an open set* $U \subset M_k$, $\mu(M_k \setminus U) < \varepsilon$, *and* ψ *has a version which is Hölder on* U.

Remark 5. *The dependence of* α *on the exponent* γ, *the asymptotics of the sequence* ε_n, *and the type of singularity are apparent from the proof.*

Remark 6. *The same regularity is forced upon solutions* ψ *to equation* (2), *equation* (3) *or equation* (4).

Proof. As in the proof of Theorem 1, choose any $\Lambda = M_k \in \mathcal{P}_0$ such that $\mu(\Lambda) > 0$ and let $(\hat{M}, \hat{f}, \hat{\mu})$ denote the natural extension of (M, f, μ). We have to check that there are sufficiently many backward paths that avoid passing too close to the singularity. We do this by using a Borel-Cantelli argument. Let $B_n = B(c; \varepsilon_n)$, then $\sum_{n\ge1} \mu(B_n) < \infty$, and hence we deduce that for $\hat{\mu}$-a.e. $\hat{x} \in \hat{M}$, there exists $N = N(\hat{x})$ such that $x_n \notin B_n$ for all $n \ge N$. Combining all these facts we obtain that for $\hat{\mu}$-a.e. backward orbit there exists $N(\hat{x})$ such that for all $n \ge N(\hat{x})$ the following hold simultaneously:

$$x_0 \in \Lambda \text{ and } f^n(\mathcal{P}_n[x_n]) = \Lambda. \tag{11}$$

$$\operatorname{diam}(f^{-n}(\Lambda) \cap \mathcal{P}_n[x_n]) \le \frac{1}{C(\hat{x})} \lambda^{-n}. \tag{12}$$

$$\rho(x_n, c) \ge \varepsilon_n. \tag{13}$$

The last two observations show that $\varphi \mid f^{-n}(\Lambda) \cap \mathcal{P}_n[x_n]$ is a Hölder function for $n \ge N$. We now consider the Hölder properties of φ in the cases that we are interested in. Suppose for $n \ge N$ we have $y, z \in f^{-n}(\Lambda) \cap \mathcal{P}_n[x_n]$. In the case of a logarithmic singularity we have (inside $B(c;\delta)$):

$$\begin{aligned} d(\varphi(y),\varphi(z)) &\approx |\log\rho(y,c)-\log\rho(z,c)| \leq \log\left(1+\frac{\rho(y,z)}{\rho(z,c)}\right) \\ &\leq \rho(y,z)^{1-\alpha}\frac{(\lambda^{-n}/C(\hat{x}))^{\alpha}}{\rho(z,c)} \leq \frac{C(\hat{x})^{-\alpha}}{\varepsilon_n\lambda^{n\alpha}}\rho(y,z)^{1-\alpha} \end{aligned}$$

for some constant $\alpha > 0$. For $\iota > 0$ arbitrary, we then obtain the estimate

$$d(\varphi(y),\varphi(z)) \leq \tilde{C}(\hat{x})\rho(y,z)^{1-\tilde{\alpha}-\iota} \quad \text{with} \quad \tilde{\alpha} = \limsup_{n\to\infty}\frac{\log\varepsilon_n^{-1}}{n\log\lambda} < 1.$$

Outside $B(c;\delta)$, the function φ will be γ-Hölder.

Now consider the case where φ has a finite order pole. Arguing as in the case of a logarithmic singularity we obtain (inside $B(c,\delta)$):

$$\begin{aligned} d(\varphi(y),\varphi(z)) &\approx |\rho(y,c)^{-p}-\rho(z,c)^{-p}| \\ &\leq \max\{p\rho(z,c)^{-p-1}, p\rho(y,c)^{-p-1}\}\rho(y,z) \\ &\leq \frac{C(\hat{x})^{-\alpha}}{(\varepsilon_n)^{p+1}\lambda^{n\alpha}}\rho(y,z)^{1-\alpha} \end{aligned}$$

and hence for arbitrary $\iota > 0$ we obtain

$$d(\varphi(y),\varphi(z)) \leq \tilde{C}(\hat{x})\rho(y,z)^{1-\tilde{\alpha}-\iota} \quad \text{with} \quad \tilde{\alpha} = \limsup_{n\to\infty}\frac{(p+1)\log\varepsilon_n^{-1}}{n\log\lambda} < 1.$$

Outside $B(c;\delta)$, φ will be γ-Hölder.

In the case of a bounded discontinuity, the Borel-Cantelli argument is simpler, since we only have to worry about $f^{-n}(\Lambda)\cap P_n[x_n]$ intersecting $\mathcal{C}$, which is impossible by the assumption that $\mathcal{C}\cap\overline{M}_k = \emptyset$.

So we proved now that ψ has a Hölder version on $f^{-N}(\Lambda)\cap P_N[x_N]$. To show that $\psi \mid \Lambda$ has a Hölder version, we argue as Corollary 1. The fact that $\mathcal{C}$ is disjoint from each $\overline{M}_k$ implies that the version $\tilde{\psi}$ will be Hölder. $\square$

4. One-Dimensional Systems

In this section, we consider C^2 one-dimensional systems for which ρ is Euclidean distance and μ is an invariant measure which is absolutely continuous with respect to Lebesgue. The assumption that there exists a function $K(\hat{x})$ and $\lambda > 0$ such that

$$\rho(y_n,z_n) \leq K(\hat{x})\lambda^{-n}\rho(y_0,z_0)$$

for all $n \geq 0$ and $y_n, z_n \in \mathcal{P}_n[x_n]$ can be replaced by two conditions which are commonly assumed in the literature,

(1) f is C^2 and has bounded distortion uniformly over all iterates: there exists a function $K(\hat{x})$ such that

$$\frac{|Df^n(y)|}{|Df^n(z)|} \leq K(\hat{x}) \tag{14}$$

for all $n \geq 0$ and $y, z \in \mathcal{P}_n[x_n]$.

(2) Positive Lyapunov exponent, i.e. $\lambda(\mu) = \exp \int \log |Df| d\mu > 1$.

Instead of assuming (6) we may use (14) and $\lambda(\mu) > 1$ to prove the following lemma.

Lemma 2. *For $\hat{\mu}$-a.e. $\hat{x} \in \hat{M}$, there exists a constant $C(\hat{x})$ such that*

$$\operatorname{diam}(\mathcal{P}_n[x_n]) \leq C(\hat{x})\lambda^{-n} \qquad \textit{for all } n \geq 0. \tag{15}$$

Proof. The measure $\hat{\mu}$ is invariant in forward and backward time, in particular $\hat{\mu}(A) = \hat{\mu}(\hat{f}(A))$. Let $D\hat{f}^{-n}$ denote the derivative of f^{-n} restricted to an inverse branch. By the Birkhoff Ergodic Theorem,

$$\lim_{n\to\infty} \frac{1}{n} \log |D\hat{f}^{-n}(\hat{x})| \quad = \quad \int \log |D\hat{f}^{-1}| d\hat{\mu} = \int -\log |Df| d\mu = -\lambda(\mu)$$

for $\hat{\mu}$-a.e. $\hat{x}$, so $|Df^n(x_n)| \geq \frac{1}{C_0}\lambda^n$ for some $C_0 = C_0(\hat{x}) < \infty$. Using (14), we find that

$$\operatorname{diam}(\mathcal{P}_n[x_n]) \leq K \frac{|\mathcal{P}_0[x_0]|}{|Df^n(x_n)|} \leq C(\hat{x})\lambda^{-n}$$

as required. □

As a consequence of (14) for some $K > 0$

$$\frac{1}{K^2} \leq \frac{\rho(y_{i+1}, z_{i+1})\rho(\tilde{y}_0, \tilde{z}_0)}{\rho(\tilde{y}_{i+1}, \tilde{z}_{i+1})\rho(y_0, z_0)} \leq K^2 \tag{16}$$

for all $y_n, \tilde{y}_n, z_n, \tilde{z}_n \in \mathcal{P}_n[x_n]$. In dimension one it is sufficient to bound

$$\sum_{i=0}^{n-1} \| \operatorname{Ad}(\varphi_i(y_i))\| \; d(\varphi(y_{i+1}), \varphi(z_{i+1}))$$

by

$$\sum_{i=0}^{n-1} C \cdot K(\hat{x})(\mu_u)^{i+1}\lambda^{-(i+1)\alpha}\rho(y_0, z_0)^\alpha$$

in (9) of the proof of the main theorem. Using the two observations (15) and (16) the proof goes through as in Theorem 1.

4.1. **Smooth measures.** The assumption that the Jacobian J_μ is Hölder is used by e.g. Young [45], and enables us to apply the technique to Gibbs measures and equilibrium states of suitable potentials. But in the case that μ is absolutely continuous with respect to Lebesgue measure, it suffices to assume that the density $h \in L^1(\mathrm{Leb})$. Indeed, equation (10) can be derived as follows. Let $\varepsilon > 0$ and $\eta > 0$. Choose n sufficiently large that $\mathrm{Leb}(\mathcal{P}_n[x_n] \cap \{y \in \mathcal{P}_n[x_n] : d(\psi(x_n), \psi(y)) < \eta\}^c) \leq \eta \,\mathrm{Leb}(\mathcal{P}_n[x_n])$. Boundedness of distortion gives that

$$\frac{\mathrm{Leb}(f^n(\mathcal{P}_n[x_n] \cap \{y \in \mathcal{P}_n[x_n] : d(\psi(x_n), \psi(y)) < \eta\}^c))}{\mathrm{Leb}(f^n(\mathcal{P}_n[x_n]))}$$
$$\leq \mathcal{O}(1) \ \frac{\mathrm{Leb}(\mathcal{P}_n[x_n] \cap \{y \in \mathcal{P}_n[x_n] : d(\psi(x_n), \psi(y)) < \eta\}^c)}{\mathrm{Leb}(\mathcal{P}_n[x_n])}.$$

Now write

$$\begin{aligned}
\mu(\Lambda) &= \mu(f^n(\mathcal{P}_n[x_n] \cap \{y \in \mathcal{P}_n[x_n] : d(\psi(x_n), \psi(y)) < \eta\}) \\
&\quad + \mu(f^n(\mathcal{P}_n[x_n] \cap \{y \in \mathcal{P}_n[x_n] : d(\psi(x_n), \psi(y)) < \eta\}^c)) \\
&= \int_{f^n(\mathcal{P}_n[x_n] \cap \{y \in \mathcal{P}_n[x_n] : d(\psi(x_n), \psi(y)) < \eta\}} h(x)dx \\
&\quad + \int_{f^n(\mathcal{P}_n[x_n] \cap \{y \in \mathcal{P}_n[x_n] : d(\psi(x_n), \psi(y)) < \eta\}^c)} h(x)dx
\end{aligned}$$

and note that since the density $h(x) \in L^1(\mathrm{Leb})$, and $\mathrm{Leb}(f^n(\mathcal{P}_n[x_n] \cap \{y \in \mathcal{P}_n[x_n] : d(\psi(x_n), \psi(y)) < \eta\}^c)) \leq \mathcal{O}(\eta)\mathrm{Leb}(\Lambda)$ we can assume $\eta > 0$ is such that $\mu(f^n(\mathcal{P}_n[x_n] \cap \{y \in \mathcal{P}_n[x_n] : d(\psi(x_n), \psi(y)) < \varepsilon\}^c)) \leq \varepsilon\mu(\Lambda)$. Here we have used the fact that for a $L^1(\mathrm{Leb})$ function g: given $\delta_1 > 0$, there exists a $\delta_2 > 0$ so that $\mathrm{Leb}(A) < \delta_2$ implies $\int_A g < \delta_1$.

5. Refinements and Applications

5.1. **Young towers.** We will show that Theorem 1 implies Hölder regularity for measurable solutions to a broad class of cohomological equations on Young towers that arise in applications.

Suppose $T : X \to X$ is a $C^{1+\gamma}$ mapping of a Riemannian manifold X and Leb denotes Lebesgue measure. Let ρ_X denote the corresponding metric. A Young tower for T has the properties:

- There exists a set $\Lambda \subset X$, decomposed as $\Lambda = \cup_j \Lambda_j \mod \mathrm{Leb}$.
- For each j, there exists $R_j \geq 1$ such that $T^{R_j} : \Lambda_j \to \Lambda$ is bijective. Denote the *induced map* $T^{R_j}|\Lambda_j$ by F.

- The distortion is bounded, i.e. there exists $K < \infty$ such that for all $n \geq 0$
$$\left| \frac{\text{Jac } DF_x^n}{\text{Jac } DF_y^n} \right| \leq K$$
for all $x, y \in U$ and sets U on which F^n is a diffeomorphism.
- There exists $\lambda_0 > 1$ such that $\min_{|v|=1} |DF_x v|/|v| \geq \lambda_0$ for $x \in \Lambda$.

The Folklore Theorem [29] states that F has an invariant probability μ, which is equivalent to Lebesgue and the Radon-Nikodym derivative h is bounded and bounded away from 0.

The measure μ can be pulled back to a T-invariant measure ν:
$$\nu(A) = \sum_j \sum_{i=0}^{R_j - 1} \mu(T^{-i} A \cap \Lambda_j). \tag{17}$$
The measure ν is finite if and only if
$$\mathcal{R} := \int_\Lambda R \, d\mu = \sum_j R_j \mu(\Lambda_j) < \infty. \tag{18}$$

This set-up can be viewed as a Markov system as follows:

- M is the disjoint union $\sqcup_j \sqcup_{i=0}^{R_j - 1} (\Lambda_j, i)$ where each (Λ_j, i) is a copy of Λ_j. The set $\Lambda = \cup_j (\Lambda_j, 0)$ is called the *base* of the tower. Each set (Λ_j, i) is a component M_k of M.
- For the metric ρ on M, there are at least two choices. Take $(x, i) \in (\Lambda_j, i)$ and $(\tilde{x}, \tilde{i}) \in (\Lambda_{\tilde{j}}, \tilde{i})$
  - ◇ $\rho_1((x,i),(\tilde{x},\tilde{i})) = \begin{cases} \rho_X(T^i(x), T^{\tilde{i}}(\tilde{x})) & \text{if } j = \tilde{j} \text{ and } i = \tilde{i}; \\ 1 & \text{otherwise.} \end{cases}$

    This metric is induced from the metric ρ_X on X. The metric ρ_1 is used in Corollary 2.
  - ◇ $\rho_2((x,i),(\tilde{x},\tilde{i})) = \begin{cases} \rho_X(x, \tilde{x}) & \text{if } j = \tilde{j} \text{ and } i = \tilde{i}; \\ 1 & \text{otherwise.} \end{cases}$

    This metric is the *tower metric* and is induced from the metric on the base Λ.
- Define $f : M \to M$ as
$$f(x,i) = \begin{cases} (x, i+1) & \text{if } x \in \Lambda_j \text{ and } i < R_j - 1, \\ (T^{R_j} x, 0) & \text{if } x \in \Lambda_j \text{ and } i = R_j - 1. \end{cases}$$
- Extend the definition of μ from Λ to M as $\mu((A,i)) = \mu(A)$ for each measurable set $A \subset \Lambda_j$ and $0 \leq i \leq R_j - 1$.

Let $\mathcal{P}_0$ denote the partition of M into sets (Λ_j, i) and set $\mathcal{P}_n = \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P}_0)$. For each $x \in M$ let $\mathcal{P}_n[x]$ be the partition element (cylinder set) in $\mathcal{P}_n$ containing x. Let $(\hat{M}, \hat{f}, \hat{\mu})$ denote the natural extension of (M, f, μ). For $\hat{x} = (x_0, x_1, \dots, x_n, \dots)$ let $n_0 < n_1 < \dots$ denote the indices such that x_{n_i} belongs to the base.

Lemma 3. *Let (M, f, μ) be a Young tower satisfying (18). Assume one of the following three conditions:*

- $\rho = \rho_1$ *and*

$$\|DT^{R_j - k}(T^k(x))\| \geq \delta_0 \text{ for all } j, x \in \Lambda_j \text{ and } 0 \leq k < R_j. \tag{19}$$

- $\rho = \rho_2$, *or*

- $\log^+ \|Df^{-1}\| \in L^1(\mu)$, *where the derivative is taken with respect to the metric used.*

Then for $\hat{\mu}$-a.e. $\hat{x}$ there exists a number $K(\hat{x})$ and $\lambda > 1$ such that

$$\rho(y_n, z_n) < K(\hat{x}) \lambda^{-n} \rho(y_0, z_0)$$

for all $n \geq 0$ and $y_n, z_n \in \mathcal{P}_n[x_n]$.

Proof. Write $R(x) = R_j$ for $x \in \Lambda_j$, so by Birkhoff's and Kac's Theorems $\mathcal{R}^{-1} = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} R(x_{n_i})$ for $\hat{\mu}$-a.e. $\hat{x}$. Let $\tilde{\Lambda}_i = \cup\{\Lambda_j \ : \ R_j = i\}$, so (18) gives $\sum_i i\mu(\tilde{\Lambda}_i) < \infty$. Given $\varepsilon > 0$, we have

$$\begin{aligned} \sum_k \hat{\mu}\{\hat{x} \ : \ R(x_{n_{k+1}}) > \varepsilon n_k\} &\leq \sum_k \hat{\mu}\{\hat{x} \ : \ x_{n_{k+1}} \in \cup_{i \geq \varepsilon k} \tilde{\Lambda}_i\} \\ &\leq \sum_k \sum_{i \geq \varepsilon k} \mu(\tilde{\Lambda}_i) \\ &\leq \lceil \frac{1}{\varepsilon} \rceil \sum_k k\mu(\tilde{\Lambda}_k) < \infty. \end{aligned}$$

The Borel-Cantelli Lemma therefore implies that for $\hat{\mu}$-a.e. $\hat{x}$, there is k_0 such that $n_{k+1} - n_0 \leq (1+\varepsilon)(n_k - n_0)$ for all $k \geq k_0$. Take $n_k \leq n < n_{k+1}$, then

$$\|Df^{-(n-n_k)}(x_{n_k})\| \leq \begin{cases} \delta_0^{-1} & \text{if } \rho = \rho_1 \text{ by (19);} \\ \lambda_0^{-1} < 1 & \text{if } \rho = \rho_2. \end{cases}$$

Write $B(\hat{x}) = \|Df^{-n_0}(x_0)\|$. Direct calculation gives

$$\begin{aligned}
\lim_{n\to\infty}\left(\|\prod_{i=0}^{n-1} Df^{-1}(x_i)\|\right)^{\frac{1}{n}} &\leq \lim_{k\to\infty}\left(\frac{B(\hat{x})}{\delta_0}\|\prod_{i=0}^{n-1} Df^{-1}(x_i)\|\right)^{\frac{1}{(1+\varepsilon)n_k}} \\
&= \lim_{k\to\infty}\left(\frac{B(\hat{x})}{\delta_0}\|\prod_{i=1}^{k} DF^{-1}(x_{n_i})\|\right)^{\frac{1}{n_0+(1+\varepsilon)\sum_{i=1}^{k} R(x_{n_i})}} \\
&= \lim_{k\to\infty}\left(\left(\frac{B(\hat{x})}{\delta_0}\|\prod_{i=1}^{k} DF^{-1}(x_{n_i})\|\right)^{\frac{1}{k}}\right)^{\frac{k}{n_0+(1+\varepsilon)\sum_{i=1}^{k} R(x_{n_i})}} \\
&\leq \lambda_0^{-\frac{1}{(1+\varepsilon)\mathcal{R}}}. \qquad (20)
\end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, we get $\lim_{n\to\infty}\left(\|\prod_{i=0}^{n-1} Df^{-1}(x_i)\|\right)^{\frac{1}{n}} \leq \lambda_0^{-\frac{1}{\mathcal{R}}}$ as well.

Alternatively, assume that $\log^+\|Df^{-1}\| \in L^1(\mu)$. The estimate of (20) holds for the sequence (n_k). Oseledec's Multiplicative Theorem then implies that $\lim_{n\to\infty}\left(\|\prod_{i=0}^{n-1} Df^{-1}(x_i)\|\right)^{\frac{1}{n}}$ exists $\hat{\mu}$-a.e. $\hat{x}$. This limit equals the limit along the subsequence, so

$$\lim_{n\to\infty}\left(\|\prod_{i=0}^{n-1} Df^{-1}(x_i)\|\right)^{\frac{1}{n}} \leq \lambda_0^{\frac{1}{\mathcal{R}}}.$$

Since these estimates hold uniformly over $y_i, z_i \in \mathcal{P}_i[x_i]$, we obtain $\rho(y_n, z_n) < K(\hat{x})\lambda^{-n}\rho(y_0, z_0)$ for $\lambda = \lambda_0^{\frac{1}{\mathcal{R}}}$. □

Remark 7. *The condition* $\log^+\|Df^{-1}\| \in L^1(\mu)$ *relates to the reference measure* m *on* X *as follows: if* μ *has density* h*, and* $h \in L^{1+\varepsilon}(m)$ *for some* $\varepsilon > 0$*, then by the Hölder inequality,* $\log^+\|Df^{-1}\| \in L^q(m)$ *for* $q = \frac{1+\varepsilon}{\varepsilon}$ *implies that* $\log^+\|Df^{-1}\| \in L^1(\mu)$.

Theorem 3. *Let* (M, f, μ) *be a Young tower with base* Λ*, satisfying* (18)*. Assume that* $\log^+\|Df^{-1}\| \in L^1(\mu)$ *or that* (19) *holds. Let* $\varphi : M \to G$ *be a Lie group valued* α*-Hölder observable and define*

$$\mu_u := \lim_{n\to\infty}\left(\sup_{(x,0)\in M}\|\operatorname{Ad}(\varphi(f^{n-1}(x,0)\ldots\varphi(f(x,0))\varphi((x,0))\|\right)^{\frac{1}{n}}.$$

If $1 \leq \mu_u < \lambda_0^{\frac{\alpha}{\mathcal{R}}}$*, then any* μ*-measurable solution to the cohomological equation*

$$\varphi = (\psi \circ f)\psi^{-1} \qquad \mu\text{-a.e.}$$

has a version which is α*-Hölder on* Λ.

Remark 8. *If G is Abelian, compact or nilpotent then $\mu_u = 1$ and the spectral condition $1 \leq \mu_u < \lambda_0^{\frac{\alpha}{\mathcal{R}}}$ is automatically satisfied.*

Remark 9. *For any n there exists an α-Hölder version on $\cup_{j=0}^{n} f^j(\Lambda)$ but the Hölder constant may increase with n.*

Remark 10. *The same result holds for solutions ψ to equation (2), equation (3) or equation (4).*

Proof. First observe that μ is an ergodic f-invariant measure. If $\rho = \rho_2$, and $x \in \Lambda_i$, then $Df(f^i(x)) = \mathrm{Id}$ if $i < R_i - 1$ and $Df(f^{R_i(x)}) = DF(x)$. If $\rho = \rho_1$, then $Df(f^i(x)) = DT(T^i(x))$. In either case, $Df(f^{R_i(x)}) = DF(x)$.

A computation similar to (20) yields that for μ-a.e. $x \in \Lambda$:

$$\lambda(\mu) := \lim_{n\to\infty} \left(\| \prod_{i=0}^{n-1} Df(f^i(x,0)) \| \right)^{\frac{1}{n}} = \lambda_0^{\frac{1}{\mathcal{R}}}. \tag{21}$$

Therefore the assumption $\mu_u < \lambda_0^{\frac{\alpha}{\mathcal{R}}}$ implies partial hyperbolicity (PH). Lemma 3 shows that condition (6) holds. Hence the proof of Theorem 1 applies and we obtain a Hölder version on each (Λ_j, i). Using the argument from Corollary 1 we obtain Livšic regularity on the base Λ as well. $\square$

Corollary 2. *Suppose (T, X) is modelled by a Young tower (M, f, μ) over base $\Lambda \subset X$, satisfying (18). Assume that $\log^+ \|Df^{-1}\| \in L^1(\mu)$ or that (19) holds. Let ν be the T invariant ergodic pulled back measure, as in equation (17). Let $\tilde{\varphi} : X \to G$ be α-Hölder and satisfy $\mu_u < \lambda_0^{\frac{\alpha}{\mathcal{R}}}$. Then for any $0 \leq k \leq \mathcal{R}_j$, any ν-measurable solution to the cohomological equation $\tilde{\varphi} = (\tilde{\psi} \circ T)\tilde{\psi}^{-1}$ has a version which is Hölder on $T^k(\Lambda_j)$.*

Proof. Suppose $\tilde{\varphi} : X \to G$ is Hölder and $\tilde{\varphi} = (\tilde{\psi} \circ T)\tilde{\psi}^{-1}$, ν-a.e. Let $\pi : M \to X$ be defined as $\pi((x, i)) = T^i(x)$, so that $T \circ \pi = \pi \circ f$. Then $\tilde{\varphi}, \tilde{\psi} : X \to G$ lift to the tower as $\varphi = \tilde{\varphi} \circ \pi$, $\psi = \tilde{\psi} \circ \pi$ and satisfy $\varphi = (\psi \circ f)\psi^{-1}$, μ-a.e. Moreover, φ is α-Hölder with respect to the metric ρ_1. Since derivatives $Df(f^i(x))$ on the tower agree with derivatives $DT(T^i(x))$ on (X, T),

$$\left(\| \prod_{i=0}^{n-1} Df(f^i(x) \| \right)^{\frac{1}{n}} \geq \lambda_0^{\frac{\alpha}{\mathcal{R}}} > \mu_u$$

for μ-a.e. x, verifying (PH) on (M, f, μ).

Now Theorem 3 gives α-Hölder (with respect to ρ_1) solution ψ' to the cohomological equation $\varphi = (\psi' \circ T)\psi'^{-1}$, such that $\psi' = \psi$ μ-a.e. Since ψ'

takes the same value on every point of $\pi^{-1}(x)$ for ν-a.e. x the projection $\tilde{\psi}' = \psi \circ \pi$ is well-defined, α-Hölder and satisfies $\tilde{\varphi} = (\tilde{\psi}' \circ T)\tilde{\psi}'^{-1}$, ν-a.e. $\square$

Remark 11. *A priori, a larger class of observables $\varphi : M \to G$ is Hölder with respect to the tower metric ρ_2 than with respect to ρ_1. For metric ρ_2, however, the projection $\pi : M \to X$, $(x, i) \mapsto T^i(x)$ need not preserve the Hölderness of solutions of the cohomologous equation on (M, f). If the projection π preserves continuity or is only used on the base Λ, this may still suffice for applications.*

5.2. **The Manneville-Pomeau family.** In the previous results, the assumptions that μ is finite and/or has positive Lyapunov exponents can be weakened for some group extensions.

For the Manneville-Pomeau family the Jacobian J_μ of μ is Hölder: i.e. there exists C and $\gamma \in (0, 1]$ such that

$$\left| \frac{J_\mu(x)}{J_\mu(y)} - 1 \right| \leq C \cdot \rho(f(x), f(y))^\gamma. \tag{22}$$

To prove Lemma 1, assumption (6) can be weakened to

$$\sum_{n \geq 0} \rho(y_n, z_n)^\gamma < K(\hat{x})\rho(y_0, z_0)^\gamma, \tag{23}$$

where $\gamma > 0$ is the Hölder exponent in (8). In this case $\lambda(\mu)$ can be 0, and (PH) fails, but if G is Abelian or compact, or if G is nilpotent with a dominated growth rate so that for inverse branches

$$\sum_{n \geq 0} \| \mathrm{Ad}(\varphi_n(y_0)) \| \; d(\varphi(y_{n+1}), \varphi(x_{n+1}))^\alpha < \infty, \tag{24}$$

then the estimates (9) hold. Furthermore, the pulled back measure ν can be at most σ-finite if (18) fails. According to equation (21), this implies that $\lambda(\mu) \leq 1$.

This scenario is found in the well-known Manneville-Pomeau maps. This is a family of maps on $[0, 1]$ which have a neutral fixed point (where we take ρ to be Euclidean distance), parameterized by parameter $p \in (0, \infty)$. For $p \geq 1$ these maps admit a σ-finite absolutely continuous invariant measure. However the measure is not a probability measure, and the map has zero Lyapunov exponents for Lebesgue almost all initial conditions. We have the following regularity result in this setting:

Theorem 4. *Consider the Manneville-Pomeau map (for $p \geq 0$):*

$$T(x) = \begin{cases} x + 2^p x^{1+p} & \text{if } x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

If G is an Abelian or compact group and $\tilde{\varphi} : [0,1] \to G$ is a G-valued α-Hölder observable for $\alpha > \frac{p}{1+p}$, then any Lebesgue measurable solution to the cohomological equation $\varphi = (\psi \circ T)\psi^{-1}$ has a version which is α-Hölder on $[0,1]$.

Remark 12. *The same regularity is forced upon solutions ψ to equation (2), equation (3) or equation (4).*

Proof. For $p \in [0,1)$, T has an invariant probability $\nu \ll$ Leb, whereas ν is only σ-finite if $p \geq 1$. However, the first return map F to $[\frac{1}{2}, 1]$ has always an invariant probability $\mu \ll$ Leb, and F^n has bounded distortion, independently of n. Therefore we can use Theorem 3 provided we can show that $\varphi|(\Lambda_j, i) = \tilde{\varphi} \circ \pi^{-1}|\Lambda_j = \varphi \circ T^i|\Lambda_j$ is Hölder. This is done as follows. It is not hard to check that $T^{-n}(\frac{1}{2}) = \frac{1}{2}(pn)^{-\frac{1}{p}} + o(n^{-\frac{1}{p}})$, where T^{-n} indicates the n-th inverse of the left branch of T. Consequently, $\mathrm{diam}(T^{-n}(J)) = \mathcal{O}(n^{-\frac{1+p}{p}})$.

Even if ν is σ-finite, $\nu([\frac{1}{2}, 1]) < \infty$. Therefore the subset $\hat{M}_0 := \{\hat{x} \in \hat{M} : x_0 \in [\frac{1}{2}, 1]\}$ has finite $\hat{\mu}$-measure. Suppose that $\hat{x} \in \hat{M}_0$ is a backward orbit (chosen as in Theorem 1), and $\hat{y}, \hat{z} \in M$ are such that $y_n, z_n \in \mathcal{P}_n[x_n]$ for each n. Let $n_0 = 0$ and $n_k = \min\{n > n_{k-1} \mid x_n \in \Lambda = [\frac{1}{2}, 1]\}$. Since $|DF| \geq 2$, $\rho(y_{n_k}, z_{n_k}) \leq 2^{-k}$, and hence, using the fact that $\alpha > \frac{p}{1+p}$,

$$\begin{aligned}
\sum_{n \geq 0} d(\varphi(y_n), \varphi(z_n)) &\leq \sum_{k \geq 0} \sum_{n=n_k}^{n_{k+1}-1} \mathcal{O}(1) \rho(y_n, z_n)^\alpha \\
&\leq \sum_{k \geq 0} \rho(y_0, z_0)^\alpha 2^{-\alpha k} \sum_{n=0}^{n_{k+1}-n_k-1} \mathcal{O}(1) \mathrm{diam}(T^{-n}([\frac{1}{2}, 1]))^\alpha \\
&\leq \rho(y_0, z_0)^\alpha \sum_{k \geq 0} 2^{-\alpha k} \sum_{n=0}^{n_{k+1}-n_k-1} \mathcal{O}(1) n^{-\alpha \frac{1+p}{p}} \\
&\leq \mathcal{O}(1) \rho(y_0, z_0)^\alpha.
\end{aligned}$$

This calculation replaces (9) in the proof of Theorem 1. Continuing the proof as in Theorem 1, we get that ψ has an α-Hölder version on $[\frac{1}{2}, 1]$. Because $T : [\frac{1}{2}, 1] \to [0,1]$ is smooth and using the cohomological equation

$\psi \circ T(x) = \varphi(x) \cdot \psi(x)$, it follows that ψ has an α-Hölder version on $[0,1]$, cf. Corollary 1. $\square$

Remark 13. *The arguments above can also be used to establish Hölder regularity for nilpotent groups G, provided (24) holds. Having $\mu_u = 1$ alone is not enough to establish Livšic regularity, but if $\| \operatorname{Ad}(\varphi_n(y_0))\|$ grows at a rate 'dominated' by the polynomial contraction of $d(\varphi(y_{n+1}), \varphi(x_{n+1}))^\alpha$, which in our case is $\mathcal{O}(n^{-\frac{\alpha(1+p)}{p}})$, then the conclusion of Theorem 4 will remain valid.*

5.3. **Interval Maps with Critical Points.** So far, the Markov systems used in the examples were Young towers, even though we allowed a σ-finite measure for the Manneville-Pomeau map. Theorem 1 also applies to different kinds of Markov systems. In this subsection we discuss the consequences of the theory to systems modelled by so-called Hofbauer towers. The metric ρ is Euclidean distance in this section.

An interval map $T : [0,1] \to [0,1]$ is called piecewise continuous (piecewise C^r) if there exists a finite set of points $0 = a_0 < a_1 < \cdots < a_{k-1} < a_k = 1$ such that $T|(a_{i-1}, a_i)$ has a continuous (C^r) extension to $[a_{i-1}, a_i]$. We call $a_0, \dots, a_k$ the *critical* points of T and denote this set by $\mathcal{C}$. Let $\mathcal{P}_1$ denote the partition $\{[a_0, a_1], \dots, [a_{k-1}, a_k]\}$, and $\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{P}_1)$ be the partition into n-cylinders. For $x \in I \setminus \cup_{i=0}^{n-1} T^{-i}(\mathcal{C})$, let $\mathcal{P}_n[x]$ denote the n-cylinder containing x.

Due to the presence of critical points, T^{-1} need not be differentiable at boundary points of $T(P)$, $P \in \mathcal{P}_1$, and hence boundedness of distortion (i.e. condition (14)) cannot be realized globally. For this reason, we call J a *core* interval if it is compactly contained in $T^n(P)$ for some $n \geq 1$ and $P \in \mathcal{P}_n$. For example, if $T(x) = 1 - ax^2$ is a non-renormalizable unimodal map, then any interval compactly contained in $[T^2(0), T(0)]$ is a core interval.

Theorem 5. *Let T be a piecewise C^3 interval map onto the unit interval with negative Schwarzian derivative. Let ν be a T-invariant probability measure such that $\int \log |T'| \, d\nu > 0$. Assume that the Jacobian J_ν is Hölder or $\nu \ll Leb$. Let φ be a Lie group valued piecewise α-Hölder observable, with discontinuities (if any) only at the points a_i, and satisfying the partial hyperbolicity condition (PH). If $\varphi = (\psi \circ T)\psi^{-1}$ for some ν-measurable function, then for every* core *interval J with $\nu(J) > 0$, $\psi|J$ has an α-Hölder version.*

Remark 14. *Assume that the core interval J is compactly contained in $T^n(P)$ for some $P \in \mathcal{P}_n$, and $n \geq 1$. Although the Hölder exponent is independent of J, the Hölder coefficient in the Hölder property of the Hölder version of $\psi|J$ depends on* $\min\{\frac{\text{diam}(J)}{\text{diam}(L)}, \frac{\text{diam}(J)}{\text{diam}(R)}\}$*, where L and R are the components of $T^n(P)\setminus J$. As this minimum tends to* 0*, the Hölder coefficient tends to infinity.*

Remark 15. *The assumption that T is onto is not a severe restriction. Since ν is assumed to have a positive Lyapunov exponent, ν cannot be supported on a non-repelling periodic orbit, so $\cap_j T^j([0,1])$ is a non-trivial interval. For the same reason, ν cannot be supported on a wandering interval, i.e. an interval W such that $T^n(W) \cap T^m(W) = \emptyset$ for all $n \neq m \geq 0$. By restricting and rescaling, we can assume that* $T : [0,1] \to [0,1]$ *is onto.*

Remark 16. *The same result holds for solutions ψ to equation* (2)*, equation* (3) *or equation* (4)*.*

Proof. We construct a Markov system, introduced by Hofbauer [19] as the *canonical Markov extension* and sometimes called a *Hofbauer tower*. This is the system (M, f), where M is a disjoint union of closed intervals. We call $B = [0,1]$ the *base of the tower*. Then

$$M = B \sqcup \left(\sqcup_{n\geq 1} \sqcup_{P\in\mathcal{P}_n} \overline{T^n(P)} \right) / \sim$$

where $T^n(P) \sim T^{n'}(P')$ if they are the same interval. Let $\pi : M \to [0,1]$ be the natural projection. The action f is defined on M as follows. If $x \in M$ belongs to the component D, then

$$f(x) = \pi^{-1}(T(\pi(x))) \cap \tilde{D},$$

where component $\tilde{D} := \overline{T(D \cap \mathcal{P}_1[\pi(x)]))}$ is again a component of M. Obviously (M, f) is Markov and $T \circ \pi = \pi \circ f$. Due to the Markov property, the following holds for any component D of M:

$$T^n(\mathcal{P}_n[x]) = \pi(D) \text{ if and only if } f^n(\pi^{-1}(x) \cap B) \in D.$$

If ν is T-invariant, then we can construct a measure μ as follows. Let μ_0 be the measure ν lifted to the level B and set $\mu_n = \frac{1}{n+1}\sum_{i=0}^{n} \mu_0 \circ f^{-i}$. Clearly $\nu = \mu_n \circ \pi^{-1}$ for each n. As shown in [24], μ_n converges vaguely. We call the limit measure μ. If ν is ergodic, then μ is either a probability measure on M, in which case we call μ *liftable*, or it is identically 0 on M. In this case the mass "has escaped to infinity". Keller's result [24, Theorem 3] states if

$\int \log |T'| \, d\nu > 0$, then ν is liftable to an invariant measure μ on the Markov extension. Moreover, μ is ergodic if ν is.

If J is a core interval and $\nu(J) > 0$, then there is some level $D \in M$ compactly containing a lifted copy $\tilde{J} := D \cap \pi^{-1}(J)$, and $\mu(\tilde{J}) > 0$. Let $\delta > 0$ be such that D contains a $\delta|J|$-neighbourhood of $\tilde{J}$. Using negative Schwarzian derivative and the Koebe principle, see [30], we find that for every $x \in M$ with $f^n(x) \in \tilde{J}$, that f^n has bounded distortion on the component of $f^{-n}(\tilde{J})$ containing x. In fact, the distortion depends only on δ.

Finally, we will show that $\psi|J$ has a Hölder version. $\tilde{\varphi} = \varphi \circ \pi$ is an observable on the Markov extension. The coboundary ψ lifts to a coboundary $\tilde{\psi} = \psi \circ \pi$. Therefore we can apply Theorem 1 to it to find a version of $\tilde{\psi}$ that is α-Hölder. Projecting it back to the interval, we find the desired α-Hölder version of ψ. (Note that since $\tilde{\psi}$ takes the same value on every point in $\pi^{-1}(x)$, we find that the Hölder version of ψ does not depend on the level $D \in M$ that we lift J to.) $\square$

Let $T : I \to I$ be a C^3 S-multimodal map having critical set $\mathcal{C}$. Assume each critical point $c \in \mathcal{C}$ has order ℓ_c, $1 < \ell_c < \infty$. We assume for simplicity that T is locally eventually onto, i.e., there is some interval I such that for every non-degenerate subinterval $U \subset I$, $T^n(U) = I$ for some $n \geq 0$. This excludes that g is renormalizable, or has a non-expanding periodic orbit. (Also wandering intervals are excluded, but this is already a corollary of the smoothness, see [30].) In this case the T-invariant measure ν that we will be considering is supported on I.

Theorem 6. *Assume $T : I \to I$ is C^3 multimodal with negative Schwarzian derivative and non-flat critical points. Write $c_n = T^n(c)$ and $\ell_{\max} = \max\{\ell_c \ : \ c \in \mathcal{C}\}$. Assume that T satisfies the summability condition*

$$\sum_{c \in \mathcal{C}} \sum_{k \geq 1} |DT^{k-1}(c_1)|^{-1/\ell_{\max}} < \infty, \tag{25}$$

and hence possesses an acip ν (cf. [36, 11]). Let φ be a piecewise α-Hölder $L^1(\nu)$ observable, with discontinuities of types (1)-(3) of Subsection 3.1, at critical points only. If $\varphi = \psi \circ T - \psi$ for some ν-measurable function, then for every core interval J, $\psi|J$ is $\tilde{\alpha}$-Hölder for any $\tilde{\alpha} \in (0, \alpha)$.

Proof. Assume as above that T is locally eventually onto. We build a Markov extension (M, f) as in Theorem 5. Since T admits an acip ν (with necessarily positive Lyapunov exponent, cf. [23] and also [9]), it can be lifted to an acip μ on (M, f). Let D be any level in M that compactly contains a lifted copy $\tilde{J} = D \cap \pi^{-1}(J)$ of J and such that $\nu(J) > 0$.

Let $(\hat{M}, \hat{f}, \hat{\mu})$ be the natural extension of (M, f, μ). By the Koebe principle, we have a uniform distortion bound for $f^n|\mathcal{P}_n[x_n] \cap f^{-n}(J)$ for each $x \in \tilde{J}$ (i.e. J is a core interval as introduced before).

The next thing to check is that there are sufficiently many backward paths that avoid passing close to the singularities of φ at $\mathcal{C}$. This argument is similar to the one in the proof of Theorem 2. The proof that (I, f) has an acip was given in the multimodal case in [11], based on the well-known result of Nowicki & van Strien [36]. An important estimate in [11] is that for some $C = C(T) < \infty$,

$$\nu(A) \leq C \cdot \mathrm{Leb}(A)^{1/\ell} \text{ for all measurable } A \subset I.$$

It follows that if $B_n = (c - n^{-2\ell}, c + n^{-2\ell})$, then

$$\nu(B_n) \leq C \cdot \mathrm{Leb}(B_n)^{1/\ell} \leq \frac{2C}{n^2},$$

and hence, lifted to the tower:

$$\mu(T^{-n}(\tilde{J}) \cap B_n) \leq \nu(B_n) \leq \frac{2C}{n^2}.$$

It follows from the Borel-Cantelli Lemma that for $\hat{\mu}$-a.e. $\hat{x} \in \hat{M}$, there exists $N = N(\bar{x})$ such that $x_n \notin B_n$ for all $n \geq N$. From now on, the argument is the same as in Theorem 2. $\square$

In view of questions raised in e.g. [12], we are particularly interested in the potential $\varphi := \int_I \log |T'| d\mu - \log |T'|$. This potential is smooth, except for logarithmic singularities at the critical points. Theorem 6 shows that any ν-measurable solution of the cohomological equation $\varphi = \psi - \psi \circ T$ has a version which is α-Hölder for any $\alpha \in (0, 1)$ on each interval that is compactly contained in $[c_2, c_1]$. We can apply it to the quadratic family $f_a(x) = 1 - ax^2$. It is known that for Leb-a.e. $a \in [0, 2]$, f_a has either an attracting periodic orbit, or has a positive Lyapunov exponent at the critical value, and both parameter sets have positive measure. Below we make a weaker assumption on the growth rate of derivatives along the critical orbit.

Corollary 3. *Let $f : I \to I$ be a C^3 S-unimodal map with critical order $\ell < \infty$, satisfying the summability condition $\sum_n |Df^n(c_1)|^{-1/\ell} < \infty$ (and hence possessing an acip ν). Then $\varphi = \log |f'| - \int \log |f'| d\nu$ can only be a* *measurable coboundary if there exists a periodic interval $J \subseteq I$ of period k such that $c \in J$ and $J = [f^{2k}(c), f^k(c)]$.*

Proof. First assume that f is nonrenormalizable. In this case, ψ is Hölder continuous and hence bounded on any interval compactly contained in $[c_2, c_1]$. Since φ is bounded except at c, we easily derive that ψ has to be unbounded at every forward image of the critical point. This is only possible if $\cup_{n\geq 1} f^n(c) \subset \{c_1, c_2\}$.

Next assume that f is finitely renormalizable, say $J \ni c$ is a periodic interval, $f^k(J) \subset J$, $f^k(\partial J) \subset \partial J$ where $k \geq 2$ is the period of renormalization, and $f^k|J$ is unimodal and nonrenormalizable. Then the above argument shows that $\cup_{n\geq 1} f^n(c) \cap J^\circ = \emptyset$. Therefore, $J = [f^{2k}(c), f^k(c)]$, and $f^k|J$ is conjugate to $x \mapsto 1 - 2x^2$. $\square$

In the context of parametrized S-unimodal families f_a, the conclusion of this corollary is that under the summability condition $\sum_n |Df^n(c_1)|^{-1/\ell} < \infty$, φ is not a measurable coboundary except, possibly, for countably many parameter values. If f_a is infinitely renormalizable, then the summability condition fails. If f_a is finitely renormalizable, then equivalently, there is a smallest periodic interval $J \ni c$ with $f^k(J) \subset J$. Only when the renormalized map $f^k : J \to J$ is conjugate to the full unimodal map $x \mapsto 1 - 2x^2$ (and for each $k \geq 1$ and configuration of $J, f(J), \dots, f^{k-1}(J)$, this usually holds for only one parameter), then it is possible that φ is a measurable coboundary. However, even in this situation, it seems extremely unlikely that φ is a measurable coboundary.

Example: If $f(x) = 1 - ax^2$ is the quadratic map, then $\varphi = \log|f'| - \int \log|f'|\, d\nu$ is a measurable coboundary for $a = 2$, but not for the parameter $a \approx 1.54368901$ at which f is renormalizable of period 2, and $f^3(c)$ is the orientation reversing fixed point.

Proof. For $a = 2$, then f is a Chebychev polynomial, and hence $h(x) = -\cos 2x$ conjugates T with the tent map $T(x) = \min\{2x, 2(1-x)\}$: $h \circ T = f \circ h$. It follows that $\varphi = \log|f'| - \log 2 = \psi - \psi \circ f$ for $\psi = -\log|h' \circ h^{-1}|$.

Next assume that f has a k-periodic interval J as in the proof of Corollary 3, such that $F := f^k|J$ is conjugate to $x \mapsto 1-2x^2$. Applying the cohomological equation to F, we get $\log|F'| - k\int \log|f'|\, d\nu = \psi - \psi \circ F$. If $h : [0,1] \to J$ is defined by $\psi = -\log|h' \circ h^{-1}|$, then $F \circ h = h \circ T$ for the tent map T as above. In particular, $k \int \log |f'| \, d\mu = \log 2$. By the cohomological equation, each periodic point $y \in J^\circ$ must have multiplier $\log 2$.

For $k = 2$, $F|J$ has an orientation reversing fixed point q_1, and $\{q_1, q_2 := f(q_1)\}$ is the corresponding period 2 orbit under f. We find $\log |F'(q_1)| = \log |f'(q_1) \cdot f'(q_2)| = \log 4|1 - a|$. So $\log |F'(q_1)| = \log 2$ only if $a = \frac{1}{2}$ or $\frac{3}{2}$, but neither parameter value corresponds to the required renormalizable map. $\square$

6. Livšic theorems for non-uniformly hyperbolic systems

In this section we use Young towers [45, 46] to prove measurable Livšic theorems for Lie group valued cocycles over non-uniformly hyperbolic systems. In particular we are able to prove Livšic theorems for Hénon maps [5]. These maps take the form $f(x, y) = (1 - ax^2 + y, \, bx)$, with $a \simeq 2$ and $b \simeq 0$, where for a positive Lebesgue measure subset of parameter space (a, b), it is proved that these maps admit a nontrvial attracting set with an ergodic Sinai-Ruelle-Bowen measure supported on it, see [13]. A Markov extension can be associated to such maps, as we describe below. Our results are also applicable to other non-uniformly hyperbolic systems which can be shown to admit a Young tower, see for example [3, 17, 44].

We discuss diffeomorphisms for which there exists a stable foliation. For the non-uniformly expanding case drop all references to the stable foliation. We refer to Young's original papers [45, 46] and Baladi's book [4] for more details. In a non-uniformly hyperbolic system the unstable leaves are not invariant under the return map to a reference set and this introduces some complications to the analysis. The proof in this section is based on that of Theorem 3 and Corollary 2.

Let $T : X \to X$ be a $C^{1+\varepsilon}$ diffeomorphism, where X is a compact manifold with metric ρ_X. Suppose there exists $\Lambda \subset X$ with a hyperbolic product structure [45, Definition 1] $\Lambda = \{(\cup\gamma^u) \cap (\cup\gamma^s) : \gamma^u \in \Gamma^u, \, \gamma^s \in \Gamma^s\}$, where Γ^u, Γ^s are two families of C^1 disks in X with the following properties: (i) disks in Γ^u are pairwise disjoint, and the disks in Γ^s are pairwise disjoint, (ii) every $\gamma^u \in \Gamma^u$ meets every $\gamma^s \in \Gamma^s$ in exactly one point, (iii) there exists a lower bound on the angle between γ^u and γ^s at the point of intersection, and (iv) each $\gamma^u \in \Gamma^u$ satisfies $m_\gamma(\gamma^u \cap \Lambda) > 0$, where μ_γ is the measure on γ^u induced by the Riemannian structure of X.

Under assumptions P1-P5 [45, Section 1], Young constructs a Markov extension (Young tower) (F, Δ) over $T : X \to X$ with base Λ. The set Λ is decomposed as $\Lambda = \cup_j \Lambda_j$ and there is a return function $R : \Lambda \to \mathbb{N}$, with constant value R_j on each Λ_j. Define $T^R(x) = T^{R(x)}(x)$.

$\Delta := \{(x, l) : x \in \Lambda;\ l = 0, 1, \dots, R(x)-1\}$ where $\Delta_0 = \{(x, 0) : x \in \Lambda\} = \Lambda$ in a natural identification. Define $F : \Delta \to \Delta$ as

$$F(x, i) = \begin{cases} (x, i+1) & \text{if } x \in \Lambda_j \text{ and } i < R_j - 1, \\ (T^{R_j} x, 0) & \text{if } x \in \Lambda_j \text{ and } i = R_j - 1. \end{cases}$$

The tower (F, Δ) is then reduced to an expanding map $\bar{F}^R : \bar{\Delta}_0 \longrightarrow \bar{\Delta}_0$, where $\bar{\Delta}_0$ is the quotient of Δ_0 under the equivalence relation that two points are equivalent if and only if they belong to the same local stable leaf γ^s.

We need the following properties.

(A1) There exists an F-invariant probability measure ν with conditional measures $\{\nu^\gamma\}$ on $\gamma^u \cap \Lambda$ leaves with densities $\{\rho^\gamma\}$ which satisfy: $\frac{1}{C} \leq \rho^\gamma \leq C$ for some $C > 0$, [46, Section 2].

(A2) There is a countable partition $\mathcal{P}_0$ of $\Lambda = \Delta_0$ into elements $\{\Lambda_j\}$ together with a return function $R : \Lambda \to \mathbb{N}$, with $R \mid_{\Lambda_i} = R_i$. Moreover F^{R_i} maps Λ_i bijectively onto Λ [46, Section 1.1].

(A3) There exists $K > 0$ such that if $y \in \gamma^u(x)$:

$$\frac{1}{K} \leq \frac{\|D^u(F^j x)\|}{\|D^u(F^j y)\|} \leq K,$$

for all $j = 0, \dots, s(x, y)$ [46, P4(b)]. Here D^u denotes the derivative along unstable leaves γ^u, and $s(x, y)$ is the first time n for which $F^n x$ and $F^n y$ lie in different elements of $\mathcal{P}_0$.

(A4) For $\gamma, \gamma' \in \Gamma^u$, if $\Theta : \gamma \cap \Lambda \longrightarrow \gamma' \cap \Lambda$ is defined by $\Theta(x) = \gamma^s(x) \cap \gamma'$, then Θ is absolutely continuous, and there exists a $C_1 > 0$ such that

$$\frac{d(\Theta_*^{-1} \mu_{\gamma'})}{d\mu_\gamma}(x) \leq C_1,$$

for all $x \in \gamma^u$, [46, P5(b)].

(A5) There exists $\lambda_u > 1$ such that for each $x \in \Lambda$, $|D^u F^R(x)| \geq \lambda_u$, [46, Section 3.1].

(A6) There exists $\lambda_s < 1$ such that for all $\gamma^s \in \Gamma^s$ and every x, y in the same γ^s, $\rho_X(F^j x, F^j y) \leq C \lambda_s^j$, cf. [46, P3]. Here $C > 0$ is a uniform constant.

(A7) Let $\mathcal{P}_n[x]$ denote the element of the partition $(T^R)^{-n}\mathcal{P}_0$ that contains $x \in \Lambda$. For $y \in \mathcal{P}_n[x]$ define $\tau(y) = R(y) + R(T^R y) + \ldots + R((T^R)^{n-1}y)$. Note that $\tau(y) = \tau(z)$ if $y, z \in \mathcal{P}_n[x]$. Let A_n be an element of the partition $(T^R)^{-n}\mathcal{P}_0$. Given $x_0 \in \Lambda$ and $x_{\tau_n} \in A_n$ with $T^{\tau_n} x_{\tau_n} = x_0$ define $x_i = T^{\tau_n - i}x_0$. Let $\hat{x} = (x_0, x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_n}, \ldots)$ be a point in the natural extension of $T^R : \Lambda \to \Lambda$ with corresponding invariant measure $\hat{\nu}_0$. We assume either

(1) a one-dimensional unstable manifold or
(2) for $\hat{\nu}$-a.e. $\hat{x}$ there exists $C(\hat{x})$ such that for all $z_{\tau_n}, y_{\tau_n} \in A_n$, $0 \leq i \leq \tau_n$,

$$\rho_X(y_i, z_i) \quad \leq \quad C(\hat{x})\lambda_u^{i-\tau_n}\rho_X(y_0, z_0). \tag{26}$$

Cocycle assumptions: Let $\varphi \colon X \to G$ be Hölder of exponent $\alpha > 0$, and define constants μ_u (as in Section 2) and μ_s by:

$$\begin{aligned} \mu_u &:= \lim_{n\to\infty}\left(\sup_{x\in X}\|\operatorname{Ad}(\varphi_n(x))\|\right)^{\frac{1}{n}}, \\ \mu_s &:= \lim_{n\to\infty}\left(\sup_{x\in X}\|\operatorname{Ad}(\varphi_n(x))^{-1}\|\right)^{-\frac{1}{n}}. \end{aligned}$$

where $\varphi_n(x) = \varphi(T^{n-1}x)\ldots\varphi(x)$. Assume a partial hyperbolicity condition (PH) on the group extension:

$$\lambda_s^\alpha < \mu_s \leq 1 \leq \mu_u < \tilde{\lambda}_u^\alpha \tag{PH}$$

where $\tilde{\lambda}_u = \lambda_u^{\frac{1}{R}}$.

We define:

$$\Theta_{PH} := \max\left\{\frac{\log\mu_u}{\log\tilde{\lambda}_u}, \frac{\log\mu_s}{\log\lambda_s}\right\}.$$

The assumption (PH) implies that $\Theta_{PH} < \alpha$.

Theorem 7. *Assume that (T, X, ν) is modelled by a tower over a base set $\Lambda \subset X$, with $\Lambda \subset \operatorname{supp}(\nu)$. Suppose in addition that T has a one-dimensional unstable direction or condition (26). Let $\varphi : X \to G$ be Hölder of exponent α and suppose condition (PH) holds. If $\psi(Tx) = \varphi(x)\psi(x)$ ν-a.e. for some measurable function $\psi : X \to G$, then $\psi \mid \Lambda$ is γ-Hölder for some $\gamma \in (0,1)$.*

Remark 17. *The same regularity is forced upon solutions ψ to equation* (2), *equation* (3) *or equation* (4).

Remark 18. *Condition (PH) is automatic if G is Abelian, compact or nilpotent.*

Proof. We start with a lemma tackling the stable direction.

Lemma 4. *There exists $\psi' = \psi$ ν-a.e. and ψ' is Hölder when restricted to each $\gamma^s \in \Lambda$ (with uniform constant and exponent).*

Proof. Choose a version of ψ and $\gamma^u \in \Lambda$ so that for μ_{γ^u}-a.e. $z \in \gamma^u$, $\psi(Tz) = \varphi(z)\psi(z)$. For each $z \in \gamma^u$, each $x \in \gamma^s(z) \subset \Lambda$ define

$$\psi'(x) = \lim_{n\to\infty} \varphi(x)^{-1} \ldots \varphi(T^n x)^{-1} \varphi(T^n z) \ldots \varphi(z)\psi(z).$$

By conditions (PH) and (A6), an argument similar to that of Section 3 can be used to show that ψ' restricted to each $\gamma^s(z)$ is uniformly Hölder. Furthermore $\psi'(Tx) = \varphi(x)\psi'(x)$ for ν-a.e. $x \in \Lambda$ and hence $\psi' = \alpha\psi$ ν-a.e. for some constant group element α. □

From now on we assume that ψ has the properties specified in the lemma above, namely ψ restricted to each $\gamma^s \in \Lambda$ is uniformly Hölder. Now we need only show that ψ restricted to each γ^u is Hölder since the local product structure implies in this case that ψ is Hölder on Λ. In fact, to show that ψ restricted to each γ^u is Hölder we need only show that there is a $\gamma^u \in \Lambda$ such that ψ restricted to γ^u is Hölder, since the fact that the holonomy is Hölder and ψ restricted to each γ^s is Hölder implies the result for all $\gamma^u \in \Lambda$.

Recall that $\mathcal{P}_0$ is the partition of Λ into $\{\Lambda_j\}$, and each Λ_j contains whole stable leaves γ^s. For $i \geq 1$ let $\mathcal{P}_i = \bigvee_{j=0}^{i-1} (F^R)^{-j} \mathcal{P}_0$. Refine the partition $\{\mathcal{P}_i\}$ in the stable direction by partitioning the stable manifolds into leaves of length at most 2^{-i} to form a partition $\mathcal{Q}_i$ of Λ. Partition Λ in such a way that if $A \subset B$, $A \in \mathcal{Q}_i, B \in \mathcal{P}_i$ then $\gamma^u \cap B = \gamma^u \cap A$ for each γ^u.

The σ-algebra generated by $\bigvee_i \mathcal{Q}_i$ generates the Borel σ-algebra on Λ. By the Martingale Density Theorem, given $\eta > 0$ there exists n and an element of the partition component $A_n \in \mathcal{Q}_n$ so that for some $x \in A_n$

$$\frac{\nu\{y \in A_n : d(\psi(x), \psi(y)) < \eta\}}{\nu(A_n)} > 1 - \eta.$$

For $x \in A_n$ define $\tau_n = R(x) + R(T^R x) + \ldots + R((T^R)^{n-1} x)$ and note $T^{\tau_n}(x) = (T^R)^n x$. Given $x_0 \in \Lambda$, $x_{\tau_n} \in A_n$ with $T^{\tau_n} x_{\tau_n} = x_0$ define x_i by $T^{\tau_n - i} x_{\tau_n} = x_i$. By (A1) and (A4) we may choose a portion of leaf $\gamma_n = \gamma^u \cap A_n$ and x_{τ_n} such that

$$\frac{m_{\gamma_n}\{y \in \gamma_n \cap A_n : d(\psi(x_{\tau_n}), \psi(y)) < \eta\}}{m_{\gamma_n}(\gamma_n)} > 1 - \mathcal{O}(\eta).$$

Then $\tilde{\gamma}_n := (T^R)^n \gamma_n$ is an unstable leaf which crosses Λ completely in the unstable direction. As a consequence of (A3)

$$\frac{m_{\tilde{\gamma}_n}\{y_0 \in \tilde{\gamma}_n : d(\psi(x_{\tau_n}), \psi(y_{\tau_n})) < \eta\}}{m_{\tilde{\gamma}_n}(\tilde{\gamma}_n)} > 1 - \mathcal{O}(\eta). \tag{27}$$

On $\tilde{\gamma}_n$ define a function $\Psi_n : \tilde{\gamma}_n \longrightarrow G$ by

$$\Psi_n(y_0) = \varphi_{\tau_n}(y_{\tau_n})[\varphi_{\tau_n}(x_{\tau_n})]^{-1},$$

where $\varphi_i(x_j) = \varphi(x_{j-1}) \dots \varphi(x_{i+j})$.

Take points $z_0, w_0 \in \tilde{\gamma}_n$. Then by the cohomological equation

$$\begin{aligned}
\psi(z_0) &= \varphi_{\tau_n}(z_{\tau_n})\psi(z_{\tau_n}) \\
&= \Psi_n(z_0)[\varphi_{\tau_n}(x_{\tau_n})]\psi(x_{\tau_n})\psi(x_{\tau_n})^{-1}\psi(z_{\tau_n}) \\
&= \Psi_n(z_0)\psi(x_0)\psi(x_{\tau_n})^{-1}\psi(z_{\tau_n}).
\end{aligned}$$

By the the right-invariance of the metric and the triangle inequality we have

$$\begin{aligned}
d(\psi(z_0), \psi(w_0)) &\leq d(\Psi_n(z_0)\psi(x_0)\psi(x_{\tau_n})^{-1}\psi(z_{\tau_n}), \Psi_n(z_0)\psi(x_0)) \\
&+ d(\Psi_n(z_0)\psi(x_0)\psi(x_{\tau_n})^{-1}\psi(w_{\tau_n}), \Psi_n(z_0)\psi(x_0)) \\
&+ d(\Psi_n(z_0)\psi(x_0)\psi(x_{\tau_n})^{-1}\psi(w_{\tau_n}), \Psi_n(w_0)\psi(x_0)\psi(x_{\tau_n})^{-1}\psi(w_{\tau_n}).
\end{aligned}$$

We claim that Ψ_n is Hölder on $\tilde{\gamma}_n$ with uniform Hölder constant and exponent (the uniformity is over n in the construction).

We calculate

$$\begin{aligned}
&d(\varphi_{\tau_n}(z_{\tau_n})\varphi_{\tau_n}(x_{\tau_n})^{-1}, \varphi_{\tau_n}(w_{\tau_n})\varphi_{\tau_n}(x_{\tau_n})^{-1}) = d(\varphi_{\tau_n}(z_{\tau_n}), \varphi_{\tau_n}(w_{\tau_n})) \\
&\leq \sum_{i=0}^{\tau_n - 1} d(\varphi_i(z_i)\varphi(z_{i+1})\varphi_{n-i-1}(w_{i+1}), \varphi_i(z_i)\varphi(w_{i+1})\varphi_{n-i-1}(w_{i+1})) \\
&\leq \sum_{i=0}^{\tau_n - 1} \| \operatorname{Ad}(\varphi_i(z_i))\| d(\varphi(z_{i+1}), \varphi(w_{i+1})) \\
&\leq \sum_{i=0}^{\tau_n - 1} C(\mu_u)^{i+1} \lambda_u^{-(i+1)\alpha} \rho_X(z_0, w_0)^\alpha,
\end{aligned}$$

where in passing from the second to third line, we use condition (26). Equivalently for the unstable direction we could have used the existence of the positive Lyapunov exponent $\tilde{\lambda}_u$, together with bounded distortion as in the proof of Theorem 1. The series converges uniformly because of condition (PH). Equation (27) gives

$$\frac{m_{\tilde{\gamma}_n}\{y_0 \in \tilde{\gamma}_n : d(\psi(x_0), \psi(y_0)) < C\rho_X(x_0, y_0)^\alpha\}}{m_{\tilde{\gamma}_n}(\tilde{\gamma}_n)} > 1 - \mathcal{O}(\eta). \tag{28}$$

Recall that ψ restricted to each stable leaf in Λ is uniformly Hölder. The holonomy map along stable leaves is absolutely continuous (A4) and the density of ν with respect to Lebesgue is bounded away from zero and above by (A1). Hence [21, Proposition 19.1.1] implies that

$$\frac{\nu\{(x,y) \in \Lambda \times \Lambda : d(\psi(x), \psi(y)) < \tilde{C}\rho_X(x,y)^\alpha\}}{\nu \times \nu(\Lambda \times \Lambda)} > 1 - \mathcal{O}(\eta). \tag{29}$$

Since η is arbitrary, it follows that $\psi|_\Lambda$ has a Hölder version, thus proving Theorem 7. □

7. Appendix

Suppose $\psi : X \to G$ is a measurable function from a metric measure space into a connected finite-dimensional matrix Lie group G endowed with a right invariant metric d_G. Let $\pi_{i,j} : G \to \mathbb{R}$, $1 \leq i, j \leq 2d$ be the local coordinate chart functions. The proof of the proposition below clearly generalizes to any finite number of real-valued measurable functions and hence establishes the Martingale Convergence Theorem, since continuity is a local property.

Proposition 2. *Suppose (X, μ) is a probability space and $\{\mathcal{P}_n\}$ is an increasing sequence of partitions of X and let $\mathcal{P}_n[x]$ denote the partition element of $\mathcal{P}_n$ which contains $x \in X$. Suppose the Borel σ-algebra is generated by $\bigvee_n \mathcal{P}_n$. Let $\varphi : X \to \mathbb{R}$ be μ-measurable and $\eta > 0$. For μ-a.e. $x \in X$,*

$$\lim_{n\to\infty} \frac{\mu\{y \in \mathcal{P}_n[x] : d(\varphi(x), \varphi(y)) < \eta\}}{\mu(\mathcal{P}_n)} > 1 - \eta. \tag{30}$$

Proof. First suppose $\varphi \in L^1(\mu)$. Let $\mathcal{F}_n$ denote the σ-algebra generated by the partition $\mathcal{P}_n$. Then

$$\lim_{n\to\infty} \mathbb{E}[\varphi|\mathcal{F}_n](x) = \varphi(x), \qquad \mu - a.e.$$

by [6, Corollary 5.22]. Note $\mathbb{E}[\varphi|\mathcal{F}_n](x)$ is constant on $\mathcal{P}_n$. Choose a sequence $\{\delta_i\}$ such that $\sum_i \delta_i < \infty$. Given $\delta_i > 0$ take N_i sufficiently large that $d(\mathbb{E}[\varphi|\mathcal{F}_n](x), \varphi(x)) < \eta$ except for a set of measure at most $\delta_i\eta^2$ for all

$n \geq N_i$. For all N_i, the union U_i of the set of atoms $A \in \mathcal{P}_{N_i}$ for which

$$\frac{\mu\{y \in A : d(\mathbb{E}[\varphi|\mathcal{F}_{N_i}](y), \varphi(y)) > \eta\}}{\mu(A)} > \eta,$$

satisfies $\mu(U_i) < \eta^2 \delta_i$. By the Borel-Cantelli Lemma, μ-a.e. $x \in X$ lies in only finitely many U_i. Finally to remove the assumption that φ is integrable note that given $\varepsilon > 0$ there exists an integrable function ψ such that $\psi(x) = \varphi(x)$ except for a set of measure at most ε. An argument using approximating functions and the Borel-Cantelli Lemma gives the same result for measurable φ. □

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