# Symmetry Detectives for SBR Attractors


Martin Golubitsky
Department of Mathematics
University of Houston
Houston, TX 77204-3476
U.S.A.

Matthew Nicol
Mathematics Institute
University of Warwick
Coventry CV4 7AL
U.K.


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**Abstract**

Let $\Gamma$ be a finite group acting on $\mathbf{R}^n$ and let $x_0$ be an initial point for a $\Gamma$-equivariant map $f : \mathbf{R}^n \to \mathbf{R}^n$. The question of determining the symmetries of the $\omega$-limit set $\omega_f(x_0)$ is discussed in Barany *et. al* [1] and Dellnitz *et. al* [2]. The methods introduced in [1, 2] are based on the notion of a symmetry detective. Detectives replace the question of determining the symmetries of the set $\omega_f(x_0)$ by the easier question of determining the symmetries of a point in an associated space $W$. The detective theorem in [1] has a limitation in that its implementation tacitly assumes that $\omega_f(x_0)$ contains a point of trivial isotropy; this assumption is explicit in [2]. In this paper we extend the ideas of [1, 2] to present sufficient conditions for an equivariant polynomial $\phi : \mathbf{R}^n \to W$ to be a detective, even when the $\omega$-limit set is contained in a proper fixed-point subspace. We show that $W$ need only satisfy the conditions given in [1, 2] while the map $\phi$ has to satisfy certain conditions in addition to the ones listed in [1, 2].

We also present a density theorem for such detectives and we show that the detective for rings of $p$ coupled cells (nearest neighbor coupling) with $\mathbf{D}_p$ symmetry first given in [1] is a detective for all (SBR) attractors.

# 1 Introduction

The notion of a symmetry detective was introduced in [1, 2] to address the following question. Let $V = \mathbf{R}^n$. Given a (discrete) dynamical system $f : V \to V$ that is $\Gamma$-equivariant with respect to a finite group $\Gamma \subset \mathbf{O}(n)$ and an initial point $x_0$, how does one determine, in practice, the symmetries of the $\omega$-limit set $\omega_f(x_0)$. The idea behind symmetry detectives is to transfer the question of determining the symmetries of the set $\omega_f(x_0)$ (a difficult question) to one of determining the symmetries of a point in an auxiliary space $W$ that depends on $\Gamma$ and its action on $V$ (a simpler question).

A given symmetry detective cannot work in all cases — but theorems in [1, 2] show that detectives can work generically. An example of Gatermann [3] showed that one difficulty with the theorem in [1] is that its implementation implicitly assumes that $\omega_f(x_0)$ contains a point of trivial isotropy. This point was also noted by Tchistiakov [5]. This assumption was made explicit in [2]. It is the purpose of this paper to state and prove a detective theorem which works without the assumption of trivial isotropy. We note that Tchistiakov [5] has proved independently a similar detective theorem. Our results show that no additional conditions on the auxiliary space $W$ beyond those given in [1, 2] is needed.

It was shown in [1] that the matrix outer product $xx^t$ is a detective (for attractors containing points of trivial isotropy) for rings of $p$ identical coupled cells with nearest neighbor coupling. Such systems have dihedral $\mathbf{D}_p$ symmetry. In this paper we use our SBR detective theorem to prove that $xx^t$ is a detective (for all SBR attractors — even those contained in fixed-point subspaces). This detective has been used by Kroon and Stewart [4] to study a model for hexapodal gaits. Tchistiakov [5] has found an explicit detective for coupled systems of $p$ identical cells having all-to-all coupling. The Josephson junction system is such an example; these systems have $\mathbf{S}_p$ symmetry.

## SBR attractors

We begin by recalling the notion of a detective. To do this we need to define the terms observable, SBR measure and SBR attractor — where SBR stands for Sinai, Bowen and Ruelle. We assume that all group actions are orthogonal.

- Let $W$ be a linear space on which $\Gamma$ acts. Define an *observable* to be a $C^1$ $\Gamma$-equivariant mapping $\varphi : V \to W$.

- An *SBR measure* for a mapping $f : V \to V$ with an invariant set $A$ is an ergodic measure $\rho$ with support equal to $A$ and with the property that there exists an open neighborhood $U \supset A$ such that for every continuous function $h : V \to \mathbf{R}$ and for Lebesgue a.e. $x \in U$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} h(f^j(x)) = \int_A h d\rho. \tag{1.1}$$

- We define an *SBR attractor* to be a topologically transitive $\omega$-limit set with an SBR measure.

Typically, we apply (1.1) to observables $\varphi : V \to W$. Define

$$K_\varphi^E = \int_A \varphi d\rho. \tag{1.2}$$

One consequence of this definition is that for Lebesgue a.e. $x \in U$

$$K_\varphi^E = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \varphi(f^j(x)). \tag{1.3}$$

When necessary we use the notation $K_\varphi^E(A)$ for $K_\varphi^E$.

Suppose that the $\Gamma$-equivariant map $f$ has an SBR attractor $A$. The symmetry group of $A$ is

$$\Sigma = \{\sigma \in \Gamma : \sigma A = A\}.$$

When necessary we use the notation $\Sigma(A)$ for $\Sigma$. We denote the isotropy subgroup of the observation $K_\varphi^E \in W$ by $\Sigma_\varphi$. Since $A$ is an SBR attractor, the definition of SBR measure stated in (1.1) implies that there exists an open set $U \supset A$ such that for Lebesgue a.e. $x \in U$ the vector $K_\varphi^E$ (as computed using (1.3)) and the symmetry group $\Sigma_\varphi$ are independent of $x$.

We now define the notion of detective relevant to this paper.

**Definition 1.1** *The observable $\varphi$ is an* SBR detective *if for each SBR attractor $A$ there is an open dense subset $\mathcal{N}$ in a neighborhood of the identity in (the $C^k$ topology on)* $\mathrm{Diff}_\Gamma(V)$ *such that all $\psi \in \mathcal{N}$ satisfy*

$$\Sigma_\varphi(\psi(A)) = \Sigma(A).$$

Loosely speaking, our main theorem shows that SBR detectives form an open dense subset in the space of $\Gamma$-equivariant polynomials.

## Types of symmetry

Attractor symmetries come in two types: instantaneous and average symmetries. The element $\gamma \in \Gamma$ is an *instantaneous* symmetry of $A$ if $\gamma a = a$ for all $a \in A$; $\gamma$ is an *average* symmetry if $\gamma A = A$. Thus instantaneous symmetries fix $A$ pointwise while average symmetries need only fix $A$ setwise. We have denoted the group of average symmetries by $\Sigma$ and we denote the group of instantaneous symmetries by $T$. Note that:

(a) $T$ is a normal subgroup of $\Sigma$,

(b) $A \subset \mathrm{Fix}_V(T)$,

(c) $\Sigma' = \Sigma/T$ fixes $A$ setwise,

(d) Let $T' \equiv N_\Gamma(T)/T$. By construction $A$ has points of trivial isotropy with respect to the group action of $T'$ on $\mathrm{Fix}_V(T)$.

The proof of our SBR detective theorem works by using the same ideas that appear in the proof of the detective theorem in [2] when $A$ had trivial $\Gamma$ isotropy in the new context where $A$ has trivial $T'$ isotropy. Some care is needed in the details.

## The detective theorem of [2]

Let $W(\Gamma)$ denote the direct sum of all distinct nontrivial irreducible representations of the finite group $\Gamma$ and let $W \supset W(\Gamma)$. Theorem 3.3 [2, p. 83] states that if $\varphi : V \to W$ is a polynomial observable and the coordinate functions of $\varphi$ in the directions of $W(\Gamma)$ are nonzero, then $\varphi$ is a detective for SBR attractors containing points of trivial isotropy. The following example is a simplified version of the example of Gatermann [3] which illustrates that some restriction on the isotropy type of points in the attractors is needed to prove a general detective theorem.

Let $\Gamma = \mathbf{D}_3$ act on $\mathbf{R}^2 \cong \mathbf{C}$ in the standard way as symmetries of an equilateral triangle and on $\mathbf{R}$ by the nontrivial representation where rotations act trivially and reflections acts as multiplication by $-1$. Let $V = \mathbf{C} \oplus \mathbf{R} = W$ and note that $W(\mathbf{D}_3) = W$. Define $\varphi$ by

$$\varphi(z, r) = (z, \mathrm{Im}(z^3)).$$

Suppose an attractor $A \subset \mathrm{Fix}_V(\mathbf{Z}_3) = \{0\} \oplus \mathbf{R}$; such attractors do not have points of trivial isotropy. Note that $\varphi(A) = (0, 0)$ and there is no way to determine whether $T = \mathbf{D}_3$ or $T = \mathbf{Z}_3$ using measurements constructed from $\varphi$.

## A theorem on density of detectives

Let $\mathcal{P}_k$ denote the space of $\Gamma$-equivariant polynomial mappings of $V \to W$ of degree at most $k$. Note that $\mathcal{P}_k$ is a finite-dimensional vector space and has its natural topology.

**Theorem 1.2** *Let $\Gamma$ be a finite group acting on $V$ and let $W \supset W(\Gamma)$ be a $\Gamma$ representation. Then, for each sufficiently large $k$, there exists an open dense subset $S \subset \mathcal{P}_k$ such that each $\varphi \in S$ is an SBR detective.*

The important point is that Theorem 1.2 holds for attractors contained strictly within fixed-point subspaces as well as for attractors containing points of trivial isotropy. When an attractor is contained in a fixed-point subspace, then the support of the associated measure $\rho$ also lies within that fixed-point subspace.

We also give sufficient conditions for an equivariant polynomial from $V$ to $W$ to be an SBR detective. We begin our discussion by introducing some terminology. Note that an isotropy subgroup $T \subset \Gamma$ has the property that $\mathrm{Fix}_V(T)$ strictly contains $\mathrm{Fix}_V(\Sigma)$ whenever $\Sigma$ strictly contains $T$. The instantaneous symmetry subgroup $T$ of an attractor $A$ is the largest subgroup $T$ for which $A \subset \mathrm{Fix}_V(T)$; such subgroups $T$ are always isotropy subgroups. The average symmetries of $A$ form a subgroup that contains $T$. For each subgroup $T \subset \Gamma$ let $T' \equiv N_\Gamma(T)/T$. When $T'$ is trivial, $A$ cannot have any additional average symmetries and detecting these nonexistent symmetries is unnecessary.

Define the finite set

$$G(V) = \{T \subset \Gamma : T \text{ is an isotropy subgroup such that } T' \neq \mathbf{1}\}. \qquad (1.4)$$

It is only the subgroups in $G(V)$ that need be considered when proving that $\varphi$ is an SBR detective.

The group $T'$ acts on $\mathrm{Fix}_W(T)$; let $W'_1, \ldots, W'_\ell$ be the isotypic components of the action of $T'$ on $\mathrm{Fix}_W(T)$ corresponding to nontrivial irreducible representations. Let $\rho_i : W \to W'_i$ be orthogonal projection.

**Theorem 1.3** *Let $W \supset W(\Gamma)$. Suppose that $\varphi : V \to W$ is a $\Gamma$-equivariant polynomial with the property that for each $T \in G(V)$ and for each $i$ the projection $\rho_i \circ \varphi(\mathrm{Fix}_V(T)) \subset W'_i$ is nonzero. Then $\varphi$ is an SBR detective.*

Note that $\mathbf{1}$ is always in $G(V)$ and that the detective theorem in [2] is just the part of Theorem 1.3 that refers to the case $T = \mathbf{1}$.

## A detective for rings of coupled cells

As an application of these ideas we present a SBR detective for systems of $p$ coupled cells coupled in a ring. Such systems have $\mathbf{D}_p$ symmetry. This discussion expands on the example given in [1]. Let the state space of the coupled cell system be $V = (\mathbf{R}^m)^p$; that is, assume that each of the $p$ cells is governed by a system of equations with $m$ unknowns. The group $\mathbf{D}_p$ acts on $V$ by permuting the 'cells' $\mathbf{R}^m$. Let $W$ be the space of $mp \times mp$ symmetric matrices and let $\mathbf{D}_p$ act on $W$ by

$$\gamma \cdot w = \gamma w \gamma^t,$$

where $\gamma$ is viewed on the right as a permutation matrix.

**Proposition 1.4** *Suppose that*

$$\dot{z}_j = f(z_{j-1}, z_j, z_{j+1}, \lambda) \qquad 1 \leq j \leq p$$

*where $z_j \in \mathbf{R}^m$ is a system of coupled cells. Assume that*

$$f(a, b, c, \lambda) = f(c, b, a, \lambda) \qquad \forall a, b, c \in \mathbf{R}^m,$$

*$p \geq 3$ and $m \geq 2$. Then the $\mathbf{D}_p$-equivariant polynomial $\varphi : V \to W$ defined by*

$$\varphi(x) = xx^t$$

*is an SBR detective.*

In the next section we recall (and extend) some results from [1, 2] as well as some general results about equivariant polynomial mappings. We prove the density theorem (Theorem 1.2) in Section 3 and we discuss the sufficiency result for SBR detectives (Theorem 1.3) and the application to coupled cell systems (Proposition 1.4) in Section 4.

# 2 Background material

To prove these results we need to present several lemmas. Let $V$ and $W$ be $\Gamma$ representations and let $\Gamma_p$ denote the isotropy subgroup of a point $p$ where $p$ is either in $V$ or in $W$.

**Lemma 2.1** *Suppose that $x \in V$, $y \in W$ and $\Gamma_x \subset \Gamma_y$. Then there exists a $\Gamma$-equivariant polynomial mapping $\varphi : V \to W$ such that $\varphi(x) = y$.*

A proof, based on the Weierstrass approximation theorem, may be found in [6].

We now define integers $k(T)$ and $k$ based on the existence result in Lemma 2.1. Recall that the normalizer $N_\Gamma(T)$ is the subgroup of $\Gamma$ consisting of elements that preserve $\mathrm{Fix}_W(T)$. Recall that $T' \equiv N_\Gamma(T)/T$ and note that $T'$ acts on $\mathrm{Fix}_W(T)$. Let $W_1, \ldots, W_\ell$ be a complete set of nontrivial nonisomorphic irreducible representations of $T'$ on $\mathrm{Fix}_W(T)$ and define

$$W_T = W_1 \oplus \cdots \oplus W_\ell. \tag{2.1}$$

Let $\pi_i : W \to W_i$ be orthogonal projection and define $\pi : \mathrm{Fix}_W(T) \to W_T$ by

$$\pi = (\pi_1, \ldots, \pi_\ell). \tag{2.2}$$

Recall the definition of $G(V)$ in (1.4) and let $T \in G(V)$. Since $T$ is an isotropy subgroup, $\mathrm{Fix}_V(T)$ contains an open dense set of points with isotropy precisely $T$. Let $x \in \mathrm{Fix}_V(T)$ be a point such that $\Gamma_x = T$. Choose $y \in \mathrm{Fix}_W(T)$ such that $\pi_i(y) \neq 0$ for $i = 1, \ldots, \ell$. Since $\Gamma_y \supset T$, it follows that $\Gamma_x \subset \Gamma_y$; hence by Lemma 2.1, there exists a $\Gamma$-equivariant polynomial $\varphi$ such that $\varphi(x) = y$. Hence $\pi_i \circ \varphi \neq 0$ for

$i = 1, \ldots, \ell$. Let $k(T)$ be the smallest degree of such a polynomial mapping $\varphi$. Fix an integer $k$ such that

$$k \geq \max_{T \in G(V)} k(T).$$

Let $g : V \to W$ be a map between real vector spaces. Denote by $\langle g \rangle$ the vector subspace of $W$ generated by linear combinations of the vectors $g(x)$ where $x \in V$.

**Lemma 2.2** *Let $V$ be a $\Gamma$ representation and $W$ an irreducible $\Gamma$ representation. Let $T \subset \Gamma$ be an isotropy subgroup and let $\bar{\varphi}$ denote the restriction of $\varphi$ to* $\mathrm{Fix}_V(T)$*. Then there is an open dense set $S_T^k \subset \mathcal{P}_k$ such that if $\varphi \in S_T^k$, $\pi(\langle \bar{\varphi} \rangle) = W_T$.*

**Proof** Note that $\bar{\varphi} : \mathrm{Fix}_V(T) \to \mathrm{Fix}_W(T)$ is $T'$-equivariant. Furthermore $\langle \pi_j \circ \bar{\varphi} \rangle \cap W_j \neq 0$ for $j = 1, \ldots, \ell$. Since the subspaces $W_1, \ldots, W_\ell$ are nonisomorphic irreducible representations, it follows that $\pi(\langle \bar{\varphi} \rangle) = W_T$.

Now we prove the open and dense property that we claimed. We define $S_T$ to be the set of all $\Gamma$-equivariant $\psi \in \mathcal{P}_k$ which satisfy $\pi(\langle \bar{\psi} \rangle) = W_T$. Note that the set $S_T$ is open by definition so we need only show density of the set $S_T$ to prove the lemma.

Suppose that a $\Gamma$-equivariant polynomial $g \in \mathcal{P}_k$ has the property that $\pi_j \circ g = 0$ for some $j$. Then for $\varepsilon \neq 0$, $\pi_j \circ (g + \varepsilon \varphi) \neq 0$. Thus $S_T$ is dense in $\mathcal{P}_k$. ∎

Now define

$$S = \bigcap_{T \in G(V)} S_T^k. \tag{2.3}$$

Note that $S$ is an open dense subset of $\mathcal{P}_k$. We will show below that elements of $S$ are SBR detectives.

**Lemma 2.3** *Let $T \in G(V)$ and suppose the representation space $W \supset W(\Gamma)$. Then* $\mathrm{Fix}_W(T)$ *contains a copy of each nontrivial irreducible representation of $T'$.*

**Proof** First we establish the claim in the case that $W = L^2(\Gamma)$. Then we generalize the result to arbitrary $W$ containing all nontrivial irreducible representations of $\Gamma$. Suppose that $W = L^2(\Gamma)$. In this case

$$\mathrm{Fix}_W(T) = \{h : \Gamma \to \mathbf{R} \text{ such that } h \text{ is constant on } T \text{ cosets}\}.$$

Each such $h$ induces a map $\tilde{h} : N_\Gamma(T)/T \to \mathbf{R}$ and we get all such maps $N_\Gamma(T)/T \to \mathbf{R}$ in this way. Hence $\mathrm{Fix}_W(T) \supset L^2(T')$. Since $L^2(T')$ contains all irreducible representations of $T'$, so does $\mathrm{Fix}_W(T)$.

Now suppose that $W$ contains a copy of each nontrivial irreducible representation of $\Gamma$. Extend $W$ so that the new $W$ contains $L^2(\Gamma)$ by adding isomorphic copies of the irreducible representations which were already in $W$ (here we use the fact that $W$ contains each irreducible representation of $\Gamma$). Thus the irreducible representations of $T'$ acting on $\mathrm{Fix}_{L^2(\Gamma)}(T)$ can only be obtained by adding isomorphic copies of the irreducible representations of $T'$ acting on $\mathrm{Fix}_W(T)$. Hence $\mathrm{Fix}_W(T)$ itself contains all irreducible representations of $T'$. ∎

The proof of our main theorem uses the notion of a representation distinguishing subgroups. This idea was introduced in [1]. We say that a representation $W$ *distinguishes all subgroups* of $\Gamma$ if

$$\dim \mathrm{Fix}_W(\Delta) < \dim \mathrm{Fix}_W(\Sigma)$$

whenever $\Delta \supset \Sigma$ are distinct subgroups. Let $W(\Gamma)$ denote the direct sum of all nontrivial irreducible representations of $\Gamma$. A sufficient condition for distinguishing all subgroups of $\Gamma$ is given in [1], Theorem 4.3, p. 70.

**Lemma 2.4** *Let $\Gamma$ be a finite group and let $W \supset W(\Gamma)$. Then $W$ distinguishes all subgroups of $\Gamma$.*

We will also use a modified version of a lemma found in [1] (Lemma 5.5, p. 72) and [2]. The modification allows us to consider attractors in $\mathrm{Fix}_V(T)$. Define the smooth map $\Phi_A^\varphi : \mathrm{Diff}_\Gamma(V) \to W$ by

$$\Phi_A^\varphi(\psi) = \int_A \varphi \circ \psi \, d\rho. \tag{2.4}$$

**Lemma 2.5** *Let $\varphi : V \to W$ be a polynomial observable and suppose that $W \supset W(\Gamma)$. Then $\varphi$ is an SBR detective if for each $T \in G(V)$ and for each SBR attractor $A$ with instantaneous symmetries $T(A) = T$ there exists an open neighborhood $\mathcal{N}$ of the identity in $\mathrm{Diff}_\Gamma(V)$ such that the projections $\pi \circ \Phi_A^\varphi$ cover an open neighborhood $\mathcal{O}$ of $\pi(K_\varphi^E(A))$ in $\mathrm{Fix}_{W_T}(\Sigma')$ where $\Sigma' = \Sigma(A)/T$.*

**Proof** Choose $T \in G(V)$ and let $A \subset \mathrm{Fix}_V(T)$ be an SBR attractor with instantaneous symmetries $T$ and setwise symmetries $\Sigma'$. Recall the definitions of $W_T$ in (2.1) and $\pi$ in (2.2). Lemma 2.3 and $W \supset W(\Gamma)$ together imply that $\mathrm{Fix}_W(T)$ contains a copy of every irreducible representation of $T'$; that is, $\mathrm{Fix}_W(T) \supset W_T$. Let $V_A \equiv \bigcup \mathrm{Fix}_{W_T}(\Delta)$ where the union is taken over all subgroups $\Delta \subset T'$ containing but not equal to $\Sigma/T$. Since $W_T$ distinguishes all subgroups of $T'$, $V_A$ is a variety of codimension one or greater (by Lemma 2.4). Hence the set $\mathcal{O}' = \mathcal{O} - V_T$ is an open dense subset of $\mathcal{O}$ in $\mathrm{Fix}_{W_T}(\Sigma')$ whose closure contains $\pi(K_\varphi^E(A))$.

Since $\pi \circ \Phi_A^\varphi$ is smooth, $(\pi \circ \Phi_A^\varphi)^{-1}(\mathcal{O}') \cap U$ is an open dense subset of $U$ in $\mathrm{Diff}_\Gamma(V)$ whose closure contains the identity. It follows that for most near identity diffeomorphisms $\psi$ the observations $\pi \circ \Phi_A^\varphi(\psi)$ are not in $V_A$ and arbitrarily close to $\pi(K_\varphi^E(A))$. This proves the lemma since those observations whose projection under $\pi$ do not lie in $V_A$ have the correct isotropy group. ∎

# 3 Proof of density of SBR detectives

In this section we prove Theorem 1.2. We show that if a $\Gamma$-equivariant polynomial $\varphi$ lies in the set $S$ defined in (2.3), then $\varphi$ satisfies the requirements of Lemma 2.5 and hence is an SBR detective. This shows that most polynomial observables of sufficiently high degree are SBR detectives.

We assume that $A$ is an SBR attractor for $f$ and that $T = T(A)$. (It follows that $T$ is an isotropy subgroup.) As in [1] and [2] we may verify that the conditions of Lemma 2.5 hold if the linearization of $\Phi_A^\varphi$, see (2.4),

$$\pi \circ L_A^\varphi : C^1(V, V) \to W,$$

is onto $\mathrm{Fix}_{W_T}(\Sigma')$. This step is an application of the implicit function theorem.

Suppose $\varphi \in S$. We show that for an open dense set of $\Gamma$-equivariant diffeomorphisms $\psi$ close to the identity, the corresponding linear maps $\pi \circ L_A^{\varphi \cdot \psi}$ are onto $\mathrm{Fix}_{W_T}(\Sigma')$. Thus by Lemma 2.5 for an open dense set of near identity $\Gamma$-equivariant diffeomorphisms $\psi$ the groups $\Sigma_\varphi(\psi(A))$ and $\Sigma(A)$ are identical. Thus $\varphi$ is an SBR detective.

We now compute the derivative $L_A^\varphi$. Let $\psi_t$ be a smooth one-parameter family of $C^1$ $\Gamma$-equivariant diffeomorphisms of $V$ with $\psi_0(x) = x$ and let $X = \frac{d}{dt}\psi_t\Big|_{t=0}$. Then

$$\begin{aligned} L_A^\varphi(X) &= \frac{d}{dt}\int_A \varphi\circ\psi_t d\rho\Big|_{t=0} \\ &= \int_A D\varphi(X)d\rho. \end{aligned}$$

Note that $\pi\circ L_A^\varphi(X)$ is the derivative of the mapping $\pi\circ\Phi_A^\varphi$ at $\psi$. Note that $\pi\circ L_A^\varphi(X) \in \mathrm{Fix}_{W_T}(\Sigma')$. If $\pi\circ L_A^\varphi$ is onto $\mathrm{Fix}_{W_T}(\Sigma')$, then the implicit function theorem and Lemma 2.5 imply that $\varphi$ is an SBR detective.

There are four steps in the proof that $\pi\circ L_A^\varphi$ is onto $\mathrm{Fix}_{W_T}(\Sigma')$. They differ in only minor details from the corresponding steps given in [2].

1) Thicken $A$ to

$$A_\varepsilon = \{x \in \mathrm{Fix}_V(T) : d(x, A) < \varepsilon\}$$

where $\varepsilon > 0$ is chosen sufficiently small so that the symmetry group of $A_\varepsilon$ is the same as the symmetry group of $A$. Define the vector space $H_{\varphi,\varepsilon} \subset W_T$ by

$$H_{\varphi,\varepsilon} = \text{ span } \{\pi\circ(D\varphi)_x(X(x))\}$$

where $x \in A_\varepsilon$ and $X \in C^1(V, V)$. We also define the vector space $H_\varphi$ by

$$H_\varphi = \text{ span } \{\pi\circ(D\varphi)_x(X(x))\}$$

where $x \in A$ and $X \in C^1(V, V)$.

We claim that $H_{\varphi,\varepsilon} = W_T$. We begin by noting that if $g : \mathbf{R}^p \to \mathbf{R}^m$ is smooth and if the images $(Dg)_x X(x)$ all lie in a proper subspace of $\mathbf{R}^m$, then, modulo a fixed constant vector, the image of $g$ also lies in that subspace. Applying the first comment to $g = \pi\circ\varphi|A_\varepsilon$, it follows that if the linear subspace $H_{\varphi,\varepsilon}$ is a proper subspace of $W_T$, then $\pi(\langle\varphi\rangle)$ must lie in a proper subset of $W_T$, contradicting the assumption that $\varphi \in S$ (and hence $\pi(\langle\varphi\rangle) = W_T$).

2) Since $H_{\varphi,\varepsilon} = W_T$, we may choose a finite number of points $x_i \in A_\varepsilon$ and a finite number of vector fields $X_i \in C^1(V, V)$ such that the set of vectors $\pi\circ(D\varphi)_{x_i}(X_i(x_i))$ is a basis for $W_T$. By continuity this basis property holds for $y_i \in A_\varepsilon$ sufficiently close to $x_i$.

3) Let $P : W \to \mathrm{Fix}_W(\Sigma)$ be the orthogonal projection. In this step our goal is to show that the image of $L_A^\varphi$ is onto $H_\varphi^P$, where

$$H_\varphi^P = \text{ span } \{P\circ(D\varphi)_x(X(x))\},$$

for $x \in A$ and $X \in C^1(V,V)$. This observation is important in our context since it implies that the image of $\pi\circ L_A^\varphi$ equals $P\circ H_\varphi$. Now we duplicate the proof given in [2]. The image of $L_A^\varphi$ is contained in $H_\varphi$, so we need only prove the reverse inclusion. this is done with the aid of the trace formula [1], which gives an explicit formula for the projection $P$ defined by:

$$P(v) = \frac{1}{|\Sigma|}\sum_{\sigma\in\Sigma}\sigma(v). \tag{3.1}$$

Choose points $z_i \in A$ and vector fields $X_i \in C^1(V,V)$ so that $\{P\circ(D\varphi)_{z_i}(X_i(z_i))\}$ is a basis for $H_\varphi^P$. We now show that we may approximate each vector $P\circ(D\varphi)_{z_i}(X_i(z_i))$ arbitrarily closely by a vector in the image of $L_A^\varphi$ and, hence by linearity, we have that the image of $L_A^\varphi$ contains $H_\varphi$. For concreteness choose the vector $P\circ(D\varphi)_{z_1}(X_1(z_1))$. Let $B_\varepsilon(z_1)$ be a small ball centered at $z_1$. Let $X$ be a vector field such that

$$X(z) = \begin{cases} \frac{1}{|\Sigma|\rho(B_\varepsilon(z_1))}X_1(z_1) & \text{for } z \in B_\varepsilon(z_1), \\ 0 & \text{off of a slightly larger set.}\end{cases}$$

Use the group action of $\Gamma$ to extend $X$ to a $\Gamma$-equivariant vector field on $V$ supported on the balls $\gamma B_\varepsilon(z_1)$ (which we can assume are either disjoint or equal if $\varepsilon$ is small enough). Then,

$$\begin{aligned} L_A^\varphi(X) &= \int_A D\varphi(X)d\rho \\ &\approx \sum_{\sigma\in\Sigma}\int_{B_\varepsilon(\sigma z_1)}\frac{1}{|\Sigma|\rho(B_\varepsilon(\sigma z_1))}(D\varphi)_x(X_1(x))d\rho \\ &\underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{|\Sigma|}\sum_{\sigma\in\Sigma}(D\varphi)_{\sigma z_1}(X_1(\sigma z_1)) \\ &= P\circ(D\varphi)_{z_1}(X_1(z_1)) \quad \text{by (3.1).}\end{aligned} \tag{3.2}$$

Thus the image of $L_A^\varphi$ equals $H_\varphi^P$ and so the image of $\pi\circ L_A^\varphi$ equals $H_\varphi$. Note that in the last equality we needed to use the fact that $\rho(B_\varepsilon(\sigma x)) = \rho(B_\varepsilon(x))$, which follows from the $\Sigma$-invariance of SBR measures. See [2].

There is an error in (3.2) due to the truncation of the vector field $X$ outside the ball $B(z_1, \varepsilon)$, which we have ignored for ease of exposition. It is easy to see that this error can be controlled and we omit the proof.

4) Recall that $H_{\varphi,\varepsilon}$ equals $W_T$. Choose a basis for $W_T$ of the form

$$\{\pi\circ(D\varphi)_{x_i}(X_i(x_i))\}$$

where $x_i \in A_\varepsilon$ and $X_i \in C^1(V, V)$. We may assume that the $x_i$'s have disjoint orbits under $\Gamma$.

Now choose $a_i \in A$ close to $x_i$ and map $x_i \to a_i$ under a $\Gamma$-equivariant diffeomorphism $\psi$. We now define

$$\mathcal{H}_\psi = \text{ span } \{\pi\circ(D\varphi)_{\psi(x)}(X(\psi(x)))\}$$

where $x \in A$ and $X \in C^1(V, V)$. By our choice of $\psi$ the sets $\mathcal{H}_\psi$ and $W_T$ are equal. But, as shown in step 3, $P\circ\mathcal{H}_\psi$ is the same as the image of $\pi\circ L_A^{\varphi\cdot\psi}$.

Hence the image of $\pi\circ L_A^{\varphi\cdot\psi}$ is equal to $\mathrm{Fix}_{W_T}(\Sigma')$ which finishes the proof. ∎

# 4 Sufficiency for SBR detectives and an example

In this section we give sufficient conditions by which an equivariant polynomial is an SBR detective. These criteria are easily abstracted from the proof we have given for the density of SBR detectives in Theorem 1.2. Recall that $G(V)$ is the set of isotropy subgroups of $\Gamma$ consisting of those $T$ for which $T' = N_\Gamma(T)/T$ is nontrivial.

Recall that $W'_1, \dots, W'_\ell$ are the isotypic components of the action of $T'$ on $\mathrm{Fix}_W(T)$ corresponding to nontrivial irreducible representations and that $\rho_i : W \to W'_i$ is orthogonal projection.

**Proof of Theorem 1.3**: Since the image of $\rho_i\circ\varphi(\mathrm{Fix}_V(T))$ in the isotypic component $W'_i$ is nonzero, the image of $\rho_i\circ\varphi(\mathrm{Fix}_V(T))$ in at least one of the isomorphic copies of the irreducible representations that constitute $W'_i$ must be nonzero. Suppose that the projection into the irreducible representation $U_i \subset W'_i$ is nonzero. Let $\chi_i : W \to U_i$ be orthogonal projection. The linear span $< \chi_i\circ\varphi(\mathrm{Fix}_V(T)) >$ is a $\Gamma$-invariant subspace of $U_i$ which must equal $U_i$ as $U_i$ is an irreducible representation.

Thus for each $i < \rho_i \circ \varphi(\mathrm{Fix}_V(T)) >$ contains an irreducible representation $U_i'$ isomorphic to $U_i$. We may define $W_T = U_1' \oplus \cdots \oplus U_\ell'$. Hence the projection of the linear span of $\varphi$ into $W_T$ equals $W_T$ for each $T \in G(V)$.

In the proof of Theorem 1.2 we show that if $\psi : V \to W$ is an equivariant polynomial which lies in the set $S$ defined in (2.3), then $\psi$ is an SBR detective. We can choose $S$ so that $S$ consists of polynomial mappings of degree less than or equal to $k$ where $k$ is arbitrarily large. We choose $k$ to be at least the degree of $\varphi$. The previous discussion shows that $\varphi \in S$ and hence $\varphi$ is an SBR detective. ∎

**Proof of Proposition 1.4**: The representation space $W$ that we use here is the space of $mp \times mp$ real symmetric matrices where $\gamma \in \mathbf{D}_p$ acts on the matrix $w$ by $\gamma \cdot w = \gamma w \gamma^t$. It is easy to check that $\varphi(x) = xx^t$ is $\mathbf{D}_p$-equivariant with respect to this action on $W$. In [1] it is shown, using the theory of characters, that $W$ contains every nontrivial irreducible representation of $\mathbf{D}_p$ and that $\varphi$ detects attractors containing points of trivial isotropy. That is, the hypotheses of Theorem 1.3 are satisfied when $T = \mathbf{1}$.

The subgroups of $\mathbf{D}_p$ are either $\mathbf{Z}_q$ where $q \geq 2$ divides $p$ or are isomorphic to $\mathbf{D}_q$ where $q \geq 1$ divides $p$. The normalizers of these groups are:

$$\begin{aligned} N_{\mathbf{D}_p}(\mathbf{Z}_q) &= \mathbf{D}_p \\ N_{\mathbf{D}_p}(\mathbf{D}_q) &= \begin{cases} \mathbf{D}_q & \text{if } \frac{p}{q} \text{ is odd} \\ \mathbf{D}_{2q} & \text{if } \frac{p}{q} \text{ is even} \end{cases} \end{aligned}$$

We begin by considering the groups $\mathbf{Z}_q$. Note that

$$\mathrm{Fix}_V(\mathbf{Z}_q) = \{(x_1, \ldots, x_t, \ldots, x_1, \ldots, x_t) : x_1, \ldots, x_t \in \mathbf{R}^m\}, \tag{4.1}$$

where $t = \frac{p}{q}$. Note that $\mathbf{Z}_q$ is not an isotropy subgroup if either $q = p$ or $q = \frac{p}{2}$ since, in these cases the vectors in $\mathrm{Fix}_V(\mathbf{Z}_q)$ are also fixed by $\mathbf{D}_q$. So we may assume that $t \geq 3$. We show that the restricted system behaves like a system of $t$ cells coupled in a ring with $m$ equations. Begin by noting that

$$\mathbf{Z}_q' = \mathbf{D}_p / \mathbf{Z}_q \cong \mathbf{D}_t$$

and that $\mathbf{D}_t$ acts on $\mathrm{Fix}_V(\mathbf{Z}_q)$ by permuting the vectors $x_1, \ldots, x_t$. Moreover $\varphi$ restricted to $\mathrm{Fix}_V(\mathbf{Z}_q)$ is just the same quadratic map on the $t$ system as on the $p$

system. The hypotheses of Theorem 1.3 are satisfied for the groups $\mathbf{Z}_q$ using the calculations in [1].

We now consider the subgroups isomorphic to $\mathbf{D}_q$. Since $\mathbf{D}_q' = 1$ when $t = \frac{p}{q}$ is odd, $\mathbf{D}_q$ is not in $G(V)$ when $t$ is odd. It follows that $\varphi$ is an SBR detective when $p$ is odd; so we may assume that $p$ is even and that $\frac{p}{q}$ is even.

When $t = \frac{p}{q}$ is even there are two conjugacy classes of subgroups $\Delta$ isomorphic to $\mathbf{D}_q$. These classes are distinguished by whether the axes of symmetry of the reflections in $\Delta$ connect opposite vertices $\mathbf{D}_q^v$ or midpoints of opposite sides $\mathbf{D}_q^s$. In the first case we write $\Delta = \mathbf{D}_q^v$ and in the second we write $\Delta = \mathbf{D}_q^s$. Define $\kappa_v$ to be the reflection that fixes cell 1 and generates $\mathbf{D}_q^v/\mathbf{Z}_q$. We view the reflection $\kappa_v$ by its action on $\mathrm{Fix}_V(\mathbf{Z}_q)$ which, by (4.1), we can identify with the $t$ letters $x_1, \ldots, x_t$. Since $t$ is even we find that

$$\kappa_v(1, \ldots, t) = (2, t)\,(3, t-1) \cdots \left(\frac{t}{2}, \frac{t}{2} + 2\right)$$

Note that $\mathbf{D}_q^v$ is the standard $\mathbf{D}_q$ in $\mathbf{D}_p$.

Similarly, define $\kappa_s$ to be the reflection that generates $\mathbf{D}_q^s/\mathbf{Z}_q$ and interchanges $x_1$ and $x_s$. A computation shows that

$$\kappa_s(1, \ldots, t) = (1, t)\,(2, t-1) \cdots \left(\frac{t}{2}, \frac{t}{2} + 1\right).$$

Using the coordinates $x_1, \ldots, x_t$ on $\mathrm{Fix}(\mathbf{Z}_q)$, it follows that

$$\begin{aligned} \mathrm{Fix}_V(\mathbf{D}_q^v) &= \{(x_1, x_2, \ldots, x_{\frac{t}{2}}, x_{\frac{t}{2}+1}, x_{\frac{t}{2}}, \ldots, x_2)\} && (4.2) \\ \mathrm{Fix}_V(\mathbf{D}_q^s) &= \{(x_1, \ldots, x_{\frac{t}{2}}, x_{\frac{t}{2}}, \ldots, x_1)\} && (4.3) \end{aligned}$$

Note that when $t = 2$ the group $T = \mathbf{D}_q^s$ is not an isotropy subgroup as $\mathrm{Fix}_V(\mathbf{D}_q^s) = \{x_1, x_1\} = \mathrm{Fix}_V(\mathbf{D}_p)$. In all other cases the groups $T = \mathbf{D}_q^v$ and $T = \mathbf{D}_q^s$ are isotropy subgroups. In general, $\mathrm{Fix}_V(\mathbf{D}_q^v)$ is $m\left(\frac{t}{2} + 1\right)$-dimensional and $\mathrm{Fix}_V(\mathbf{D}_q^s)$ is $m\frac{t}{2}$-dimensional.

We claim that the hypotheses for Theorem 1.3 can be verified for $T \cong \mathbf{D}_q$ by direct calculation, from which it follows that $\varphi$ is an SBR detective. As a first step, let $\tau$ be the generator of $\mathbf{Z}_{2q}/\mathbf{Z}_q$. Then $\tau$ is a generator for $N_{\mathbf{D}_p}(\mathbf{D}_q)/\mathbf{D}_q$ for either type of group isomorphic to $\mathbf{D}_q$. The action of $\tau$ on $\mathrm{Fix}_V(\mathbf{Z}_q)$ is

$$\tau((1, \ldots, t) = (1, \frac{t}{2} + 1) \cdots (\frac{t}{2}, t)$$

Thus we can compute the action of the permutation $\tau$ on $\mathrm{Fix}_V(\mathbf{D}_q)$ in the coordinates implicit in (4.2) and (4.3) as

$$\begin{aligned}
\tau(1,\ldots,\frac{t}{2}+1) &= \left(1,\frac{t}{2}+1\right)\left(2,\frac{t}{2}\right)\cdots\left(\left[\frac{t+2}{4}\right],\left[\frac{t}{4}\right]+2\right) \\
\tau(1,\ldots,\frac{t}{2}) &= \left(1,\frac{t}{2}\right)\left(2,\frac{t}{2}-1\right)\cdots\left(\left[\frac{t}{4}\right],\left[\frac{t+2}{4}\right]+1\right)
\end{aligned}$$

In light of Theorem 1.3 $\varphi$ is an SBR detective if we can show that the projection of $\varphi$ into the nontrivial representation of $\mathbf{Z}_2$ on $\mathrm{Fix}_W(\mathbf{D}_q)$ is nonzero. The projection of $\varphi$ onto the nontrivial representation of $\mathbf{Z}_2$ on $\mathrm{Fix}_W(\mathbf{D}_q)$ is given by

$$x \mapsto \frac{1}{2}(1-\tau)\varphi(x),$$

where 1 is the identity element. Since $\varphi$ is $\Gamma$-equivariant this mapping is equal to

$$x \mapsto \frac{1}{2}(\varphi(x)-\varphi(\tau(x))). \tag{4.4}$$

By considering the case where $x_1$ is distinct from $x_{t+1}$ with $x_2 = \cdots = x_t = 0$ it is straightforward to compute that (4.4) is nonzero on $\mathrm{Fix}_V(\mathbf{D}_q^v)$. Similarly, by considering the case where $x_1$ is distinct from $x_t$ with $x_2 = \cdots = x_{t-1} = 0$ it is easily computed that (4.4) is nonzero on $\mathrm{Fix}_V(\mathbf{D}_q^s)$. ∎

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