Piecewise Isometries in Euclidean spaces of higher dimensions

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Abstract

We consider the behaviour of piecewise isometries in dimensions $n \geq 2$. We show that if n is odd and the system contains no orientation reversing isometries then recurrent orbits with rational coding are not expected. More precisely a prevalent set of piecewise isometries do not have recurrent points having rational coding. This implies that when all atoms are convex no periodic points exist for almost every piecewise isometry.

By contrast, if $n \geq 2$ is even then periodic points are stable for almost every piecewise isometry whose set of defining isometries are not orientation reversing. If, in addition, the defining isometries satisfy an incommensurability condition then all unbounded orbits must be irrationally coded.

1 Introduction

In the available literature concerning the dynamics of piecewise isometric systems, the study of rational cells and periodic points has been developed mainly in the context of bounded spaces (see Goetz (1996) and (2001)). The main goal in this paper is to extend the existing results to unbounded Euclidean spaces in dimensions greater than 2. Our motivation was drawn by comments in Goetz (1996) stating that the stability of periodic cells in piecewise rotations in the plane generalise to higher dimensions.

For bounded phase spaces, the existence of periodic points is equivalent to the existence of rational cells provided that the partition is convex (see Goetz (1996)). Theorem 1 provides an analogous statement for unbounded spaces. Goetz (1996) has also shown that positive measure cells in compact spaces must have rational coding. His arguments used recurrence and compactness and made no use of the Euclidean group structure of isometries. Conversely, Theorem 2 shows that for certain piecewise isometric systems in unbounded Euclidean spaces of even dimension every orbit escaping to infinity must be irrationally coded.

If n is odd, the canonical representation of elements in $\mathbf{SE}(n)$ implies that typically, rationally coded points will have unbounded orbit (Theorem 3). Theorem 1 implies as a corollary that a typical orientation-preserving piecewise isometry does not possess periodic points.

Finally, we generalise the stability result for periodic points stated in Goetz (1996) in Theorem 5. The analogous statements though, only hold for even dimensional spaces.

This paper is organised as follows. In the next Section we briefly recall important concepts from the theory of piecewise isometric systems as well as basic definitions related to the Euclidean group. We will also include an elementary discussion on recurrence properties of isometries in $\mathbb{R}^n$. Section Three contains all statements mentioned previously in this introduction and finally, Section Four includes a discussion on piecewise rotations of the Poincaré half plane.

2 Preliminaries

In this Section we recall basic concepts and some elementary facts from the theory of piecewise isometries and also give a brief discussion of properties of the Euclidean group $\mathbf{E}(n)$. In particular, we are interested in when an element of the group $\mathbf{E}(n)$ has a bounded orbit.

2.1 Definitions and Background

The structure of the Euclidean group

The Euclidean group $\mathbf{E}(n)$ is the group of all isometries of $\mathbb{R}^n$. It is the semidirect product of $\mathbf{O}(n)$, the orthogonal group and $\mathbb{R}^n$,

$$\mathbf{E}(n) = \mathbf{O}(n) \ltimes \mathbb{R}^n \ .$$

Recall that

$$\mathbf{O}(n) = \{A \in \mathbf{GL}(n, \mathbb{R}) : AA^t = Id\} \ .$$

We may identify an element $g \in \mathbf{E}(n)$ with a pair (A, v) where $A \in \mathbf{O}(n)$ and $v \in \mathbf{R}^n$. The group multiplication is given by $(A, v)(B, w) = (AB, Aw + v)$. The group $\mathbf{O}(n)$ includes both orientation reversing ($\mathsf{det}(A) = -1$) and orientation preserving ($\mathsf{det}(A) = 1$) transformations. We will always consider $\mathbf{O}(n)$ to have its standard matrix representation acting on $\mathbb{R}^n$ and denote the standard action of a matrix $A \in \mathbf{O}(n)$ on a vector $v \in \mathbb{R}^n$, by $v \mapsto Av$. The action of T on $x \in \mathbb{R}^n$ is given by

$$T(x) = Ax + v$$

where Ax is the usual matrix multiplication.

The subgroup $\mathbf{SO}(n) \subset \mathbf{O}(n)$ consists of orientation preserving transformations, defined by $\mathbf{SO}(n) := \{A \in \mathbf{O}(n) : \mathsf{det}(A) = 1\}$. The Special Euclidean group $\mathbf{SE}(n)$ is the semidirect product of $\mathbf{SO}(n)$ and $\mathbb{R}^n$.

For every element A of $\mathbf{O}(n)$ there is a orthonormal basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ for which A can be orthogonally diagonalised. If n is even this implies that

$$A = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & B_{n/2} \end{pmatrix}, \tag{1}$$

where,

$$B_i = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix} \text{ or } B_i = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

and $\theta_i \in [0, 2\pi)$. Note that the second form also includes the cases when $\theta_i = 0$ and $\theta_i = \pi$.

If n is odd, $\mathcal{B}$ can be chosen so that,

$$A = \begin{pmatrix} \widetilde{A} & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ where } \widetilde{A} \in \mathbf{O}(n-1) \ . \tag{2}$$

(See Mishina (1965) for more details on the canonical form of matrices in $\mathbf{O}(n)$.)

Denote by h the usual left and right invariant Haar measure on $\mathbf{O}(n)(\mathbf{SO}(n))$ and let μ be the n-dimensional Lebesgue measure on $\mathbb{R}^n$. It can easily be checked that $h \times \mu$ is a right and left invariant Haar measure on $\mathbf{E}(n)(\mathbf{SE}(n))$. When we refer to a full measure set we mean a set whose complement has zero measure, where the measure we refer to will be clear from context. Similarly we say that almost every (a. e.) element $T \in \mathbf{E}(n)(\mathbf{SE}(n))$ satisfies a condition $\mathcal{C}$ if the $h \times \mu$-measure of elements $g \in \mathbf{E}(n)(\mathbf{SE}(n))$ not satisfying condition $\mathcal{C}$ is zero. This notion can be extended to $\mathbf{E}(n) \times \overset{m}{\ldots} \times \mathbf{E}(n)$, $m \in \mathbb{N}$, by considering the product measure $(h \times \mu)^m := (h \times \mu) \times \overset{m}{\ldots} \times (h \times \mu)$. The groups, $\mathbf{SO}(n), \mathbf{O}(n), \mathbf{SE}(n)$ and $\mathbf{E}(n)$ are equipped with their usual Lie group topologies.

Basic concepts from Piecewise Isometries

Let $\mathbf{X} = \mathbb{R}^n$ or some connected subset of $\mathbb{R}^n$ endowed with the standard Euclidean metric $d(\cdot,\cdot)$ on $\mathbf{R}^n$ and let $\mathcal{P} = \{P_0, \ldots, P_{m-1}\}$ be a finite collection of connected open sets with piecewise smooth boundary such that:

(i) $P_i \cap P_j = \emptyset$, for $i \neq j$;

(ii) $\mathbf{X} = \overline{P}_0 \cup \ldots \cup \overline{P}_{m-1}$;

We will call $\mathcal{P}$ a partition (with slight abuse of notation) and each set P_i an atom of the partition. We call $\mathcal{D} = \cup_{k=0}^{m-1} \partial P_k$ the discontinuity set.

A piecewise isometry on $\mathbf{X}$ with partition $\mathcal{P}$ is a map $T : \mathbf{X} \backslash \mathcal{D} \to \mathbf{X}$, such that,

$$T(x) = T_i(x), \text{ if } x \in P_i \ ,$$

where T_i is an isometry defined on $\mathbf{X}$, for $i = 0, \dots, m-1$. We do not define T on $\mathcal{D}$ but to simplify notation $T(S)$ will represent $T(S \backslash \mathcal{D})$ for any set $S \subset \mathbf{X}$.

Given a partition $\mathcal{P}$ of a space $\mathbf{X}$, we denote by $PWI(\mathcal{P})$ the set of all piecewise isometries that can be defined in this way on $\mathcal{P}$. We define $\mathbf{SE}(\mathcal{P})$ to be the set of all piecewise isometries for which the maps $\{T_i\}$ are orientation-preserving. In other words $T \in \mathbf{SE}(\mathcal{P})$ if all $T_i \in \mathbf{SE}(n)$, the Special Euclidean group.

The partition $\mathcal{P}$ gives a natural way of coding the trajectories. We define $\mathbf{X}^* \subset \mathbf{X}$ to be the set $\{x \in \mathbf{X} : \forall n \geq 0, T^n(x) \in \mathcal{P}_{i_n}, \text{ for some } i_n\}$. Thus $\mathbf{X}^*$ is the set of points whose forward iterates avoid the discontinuity set.

Let $\mathcal{A}$ be the alphabet $\{0, \dots, m-1\}$. We define the coding map $\chi : \mathbf{X}^* \to \mathcal{A}^{\mathbb{N}}$ according to the following rule:

$$\chi(x) = w_0 \dots w_n \dots \ , \text{ if and only if } T^n(x) \in P_{w_n}, \text{ where } w_n \in \mathcal{A}.$$

If $x \in \mathbf{X}^*$ is such that its coding $\chi(x)$ is eventually periodic that is, if $\chi(x) = uv \dots v \dots$ where u and v are finite words on $\mathcal{A}$ then we call x a rational point. Otherwise $x \in \mathbf{X}^*$ is called an irrational point.

The original partition can be refined using the coding map. More precisely, let K_w be the set of all points $x \in \mathbf{X}^*$ with the same coding w. We will call the set K_w a cell. If all atoms of the partition are convex, open and w is rational then, K_w is convex and has open interior. If $x, y \in \mathbf{X}^*$ belong to the same cell, then for all $k \in \mathbb{N}$, $d(T^k(x), T^k(y)) = d(x, y)$.

Perturbing elements of $PWI(\mathcal{P})$

In order to consider perturbations of piecewise isometries we have to define a metric on $PWI(\mathcal{P})$. Nevertheless, our genericity results will not depend on the chosen metric.

First of all, we remark that, unlike Goetz (2001), we do not consider perturbations of the partition $\mathcal{P}$. The main reason being that our results hold for an unbounded space whereas results in Goetz (2001) are stated for compact sets. Moreover, if we perturb in the class of piecewise isometries with convex partition, it turns out that changing the atoms of the partition $\mathcal{P}$ is equivalent to conjugating the induced isometries with a translation. This can be easily visualised with piecewise rotations in the plane where conjugating with translations means moving the centre of rotation.

For a given $A \in \mathbf{O}(n)$ we will use the norm

$$\|A\|_{\mathbf{O}(n)} = \sqrt{tr(AA^t)} = \sqrt{\Sigma_{i=1}^n \Sigma_{j=1}^n a_{ij}^2} \ .$$

This allows us to define a metric in $PWI(\mathcal{P})$ the following manner,

Definition 1 *Given $T, T' \in PWI(\mathcal{P})$ where $T = \{(A_i, v_i)\}_{i=0}^{m-1}$ and $T' = \{(A'_i, v'_i)\}_{i=0}^{m-1}$ we define the distance between them by*

$$dist(T, T') = \Sigma_{i=0}^{m-1} \left\| A_i - A'_i \right\|_{\mathbf{O}(n)} + \Sigma_{i=0}^{m-1} \left\| v_i - v'_i \right\|_{\mathbb{R}^n} \ .$$

where $\|.\|_{\mathbb{R}^n}$ denotes the usual Euclidean norm.

2.2 Recurrence in $\mathbf{E}(n)$

Ashwin and Melbourne (1997) have considered the question of for which elements $g \in \mathbf{SE}(n)$ the orbit $\{g^n\}$ is bounded or unbounded. This question turns out to be equivalent to the question of the recurrence of the orbit $\{g^n(x)\}$ for some $x \in \mathbf{X}$. In this Section we extend the discussion to the group of isometries $\mathbf{E}(n)$ which will form the basis for our results on the behaviour of piecewise isometries.

Our first observation is that if $x \in \mathbb{R}^n$ and $T \in \mathbf{E}(n)$ then $T^k x = A^k x + v + \cdots + A^{k-1} v$ and hence the boundedness of the orbit of x is independent of x and an intrinsic property of $T = (A, v)$. Therefore we may classify elements of $\mathbf{E}(n)$ as being either bounded or unbounded according to whether the sequence of vectors $v_k := v + \cdots + A^{k-1} v$ is bounded or not.

Definition 2 *Given an isometry $T \in \mathbf{E}(n)$ we say that T is* ***recurrent*** *if there exists a recurrent point x under T. Otherwise, T is called a* ***nonrecurrent*** *isometry.*

Let T be a recurrent isometry. We will show (see proof of theorem 1) that the existence of a recurrent point for T is equivalent to the existence of a fixed point p. Furthermore,

$$Tp = p \Leftrightarrow (Id - A)p = v \ .$$

It is a simple consequence of the canonical form of A that the equation $(Id - A)X = v$ has a solution on X if and only if $< v, \mathsf{Fix} A >= 0$ and hence the following remark,

Remark 1 *An isometry $T = (A, v)$ is recurrent if and only if $< v, \mathsf{Fix} A >= 0$.*

Given an element $A \in \mathbf{SO}(n)$ the closure of $\{A^k x\}_{k \in \mathbb{N}}$ is diffeomorphic to a subgroup of $\mathbb{T}^n$. This follows easily by considering the canonical form of an element of $\mathbf{SO}(n)$ (see (1)).

The canonical form for an element $A \in \mathbf{O}(n)/\mathbf{SO}(n)$ has an entry of -1 in the bottom right hand corner corresponding to a reflection of the basis element b_n (see (2)). This implies that the closure of $\{A^k x\}_{k \in \mathbb{N}}$ is diffeomorphic to the union of two subgroups of $\mathbb{T}^n$ (one of the subgroups being a reflection of the other in the b_n component).

For a recurrent isometry $T \in \mathbf{E}(n)$ the corresponding statements hold since the action of T is conjugate to that of its linear part, A.

We finish this discussion with a brief summary of recurrent transformations in $\mathbf{E}(n)$.

$\mathbf{SE}(n)$, n even: If all $\theta_i \neq 0$ in the canonical representation of A (as shown in (1)) then $\mathsf{dim}(\mathsf{Fix}A) = 0$ and T is recurrent. Otherwise, $\mathsf{dim}(\mathsf{Fix}A) \geq 2$ and T is recurrent if and only if $< v, \mathsf{Fix}A >= 0$.

$\mathbf{SE}(n)$, n odd or $\mathbf{E}(n)\backslash\mathbf{SE}(n)$, n even: In this case $\mathsf{dim}(\mathsf{Fix}A) \geq 1$ and thus, T is recurrent if and only if $< v, \mathsf{Fix}A >= 0$.

$\mathbf{E}(n)\backslash\mathbf{SE}(n)$, n odd: Since $\mathsf{det}A = -1$ and n is odd there must exist an even number of occurences of the eigenvalue 1 (this number could be zero). Thus, if all $\theta_i \neq 0$ in the canonical representation of A (as in (2)) then $\mathsf{dim}(\mathsf{Fix}A) = 0$ and T is recurrent. Otherwise, $\mathsf{dim}(\mathsf{Fix}A) \geq 2$ and T is recurrent if and only if $< v, \mathsf{Fix}A >= 0$.

3 Main Statements and Results

This Section contains the main results of this paper. Note that throughout the presentation, n will denote the dimension of $\mathbf{X}$ and m the number of atoms in $\mathcal{P}$.

The following simple proposition generalises a result of Goetz (1996) to the setting of a noncompact space. We will make use of the convex hull of a set S which we denote by $conv(S)$ following standard notation in Webster (1994).

Theorem 1 *If $\mathcal{P}$ is a convex partition then there exist recurrent points with rational codings if and only if there exist periodic points.*

proof: If a periodic point exists then it is recurrent and has rational coding. Conversely, given a recurrent point x, with rational coding, let K_w be the cell with the same coding as x. Without loss of generality, we can assume that w is periodic. Let $T_w : K_w \to K_w$ denote the return map induced by K_w. We consider T_w as an element of the Euclidean group defined in the whole space $\mathbf{X}$.

We will use the following elementary lemma from Poggiaspalla (2000): If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry and S is a bounded and invariant set under T then its centre of mass is a fixed point for T.

For any isometry T, it follows easily that $T(conv(S)) = conv(T(S))$ for every set S and therefore, given an invariant set I, $conv(I)$ is a convex set which is also invariant under T. Consequently, $J = conv(\cup_{k=-\infty}^{\infty} T_w^k(x))$ is a bounded[1], convex and T_w-invariant set.

It now follows that the centre of mass of J, C_J, is well defined and it is a fixed point for T_w. From a result in Webster (1994) we can also state that C_J can be written as a convex combination of a finite number of points of $\cup_{k=-\infty}^{\infty} T_w^k(x)$. Furthermore, these points can be taken in $\cup_{k=0}^{\infty} T_w^k(x)$ simply by iterating under T_w if necessary which shows that $C_J \in conv(\cup_{k=0}^{\infty} T_w^k(x))$.

This implies that $C_J \in K_w$ and hence C_J is a periodic point for T. ∎

[1]This stems from the fact that the full orbit of x is diffeomorphic to or contained in a torus $\mathbb{T}^n$ or the union of two tori.

The existence of rational points alone does not imply the existence of periodic points, for if the return map of a rational cell is nonrecurrent, it is the case that no periodic points exist in that cell. A simple example of this is given by a piecewise isometry defined by a partition of $\mathbb{R}$ into $P_0 := \{x : x < 0\}$ and $P_1 := \{x : x > 0\}$ and

$$T(x) := \begin{cases} -x - 1 \text{ if } x \in P_1 \\ -x + 2 \text{ if } x \in P_0 \end{cases} .$$

Note that $T^2(P_1) \subset P_1$ and the return map to P_1 is given by $T_{P_1}.x = x + 3$ which is a translation and thus, every point in P_1 is rationally coded but there are no periodic points. Therefore, it is not sufficient to assume that the coding is rational to obtain periodic points.

However, in certain settings the fact that an orbit is unbounded implies that it is irrationally coded. Given a piecewise isometry defined by the set of induced isometries $\{(A_i, v_i)\}_{i=0}^{m-1}$ we say that the set $\{A_i\}_{i=0}^{m-1}$ is *incommensurate* if $\mathsf{Fix}A_w = \{0\}$ for every finite word w in the alphabet $\{0, \dots, m-1\}$.

Theorem 2 *Suppose n is even, $T \in \mathbf{SE}(\mathcal{P})$ and the rotations $\{A_i\}_{i=0}^{m-1}$ defining T are incommensurate. Then, every unbounded orbit is irrationally coded.*

Furthermore, for almost every $T \in \mathbf{SE}(\mathcal{P})$ the corresponding set of rotations $\{A_i\}_{i=0}^{m-1}$ defining T are incommensurate.

proof: Suppose x is rationally coded with preperiodic part s and periodic part $u = u_1 u_2 \dots u_p$. Take K_u to be the periodic cell corresponding to u and consider the return map $T_u = T_{u_p} \dots T_{u_1}$. Since the linear part of T_u is $A_u = A_{u_p} \dots A_{u_1}$ and by assumption $\mathsf{Fix}A_u = \{0\}$ we can conclude that $< \hat{v}, \mathsf{Fix}A_u >= 0$ independently of the vector part, $\hat{v}$, of T_u. This implies that T_u is a recurrent isometry and hence x must have bounded orbit.

Fix a word w of length l, say, and let $Z_w = \{A \in \mathbf{SO}(n)^m : \mathsf{Fix}A_w \neq \{0\}\}$. Since $\mathsf{Fix}A_w \neq \{0\}$ there must exist a vector $t \neq 0$ such that $(A_w - Id)t = 0$ which implies that $\mathsf{det}(A_w - Id) = 0$. Let $\phi_w : \mathbf{SO}(n)^m \to \mathbb{R}$ be the analytic function defined by $\phi_w(A_0, \dots, A_{m-1}) = \mathsf{det}(A_w - Id)$. Since n is even and for almost every $A \in \mathbf{SO}(n)$ we have that $\phi_w(A, \dots, A) = \mathsf{det}(A^l - Id) \neq 0$ the set Z_w is a Zariski subvariety of $\mathbf{SO}(n)^m$ of zero h^m-measure.

The assertion follows now since the set of finite words is countable. ∎

Remark 2 *In dimension 2 incommensurability is equivalent to there being no rational relations amongst the set $\{\theta_j\}$ of angles of rotation of the matrices $\{A_i\}$. We have been unable to determine a simple criterion for incommensurability of the matrices $\{A_i\}$ in even dimension $n > 2$.*

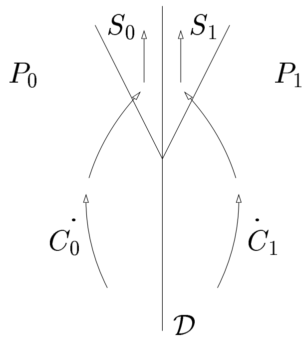
Lack of incommensurability can generate rational cells with unbounded dynamics as we now describe in the following example. Let $\alpha_0, \alpha_1 > 0$ such that $\alpha_0 + \alpha_1 = 2\pi$ and

$C_0 = (a, 0)$, $C_1 = (-b, 0)$ where $a, b > 0$. Define the atoms of $\mathcal{P}$ by $P_0 := \{(x, y) \in \mathbb{R}^2 : x < 0\}$ and $P_1 := \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Consider now the piecewise rotation defined by,

$$T(z) := \begin{cases} R_{\alpha_0}(z - C_0) + C_0 \text{ if } z \in P_0 \\ R_{\alpha_1}(z - C_1) + C_1 \text{ if } z \in P_1 \end{cases},$$

where $R_{\alpha_i} = \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix}$ for $i = 0, 1$.

Theorem 2 in Goetz (1998) implies that all orbits under T are unbounded. In Figure 1 we show a numerical simulation for a particular choice of parameters depicting the preimages of the discontinuity. The phase space decomposes into a finite number of unbounded cells each one of them settling, after a finite number of iterations, into a set of two period 2 cells on which the return map is a translation.



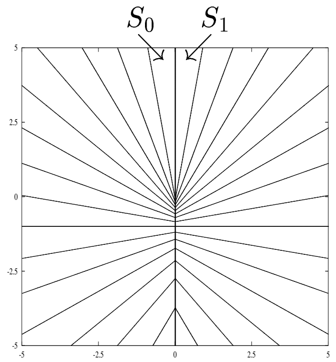


Figure 1: $a = b = 1$, $\alpha_0 = 10°$, $\alpha_1 = 350°$: (left) Points rotate until reaching one of the regions S_0 or S_1. From then on, points alternate between S_0 and S_1 such that the time-2 map is a translation. (right) 100 preimages of the discontinuity under T. White regions represent different rational cells.

Theorem 3 *If $n > 2$ is odd then a.e. piecewise isometry in $\mathbf{SE}(\mathcal{P})$ has no recurrent points with rational coding.*

proof: Assume $n > 2$ is odd. Suppose there are m atoms $\{P_i\}_{i=0}^{m-1}$ of the partition $\mathcal{P}$. An element $T \in \mathbf{SE}(\mathcal{P})$ is defined by a choice of isometries $\{(A_i, v_i)\}$, $i = 0, \ldots, m-1$. Suppose $w = w_1 \ldots w_k$ is a word of length k on $\{0, \ldots, m-1\}$. We write A_w as an abbreviation for $A_{w_1} \ldots A_{w_k}$ and similarly $T_w = T_{w_1} \ldots T_{w_k}$. T_w is of the form $(A_w, U_0 v_0 + \ldots + U_{m-1} v_{m-1})$ where U_j is a sum of words on matrices or $U_j = 0$ depending on whether or not the element T_j appears in the word T_w. The last isometry applied, T_{w_1}, distinguishes U_{w_1} as a sum of the identity matrix I and the sum of products of matrices.

A choice of matrices $\{A_0, \ldots, A_{m-1}\} \in \mathbf{SO}(n)^m$ specifies U_{w_1} and thus U_{w_1} is the image of a polynomial function, ϕ_w, on $\mathbf{SO}(n)^m$ taking values in the general linear group $\mathbf{GL}(n, \mathbf{R})$. Since $\phi_w : \mathbf{SO}(n)^m \to \mathbf{GL}(n, \mathbf{R})$ is analytic, the set of matrices Z_w in $\mathbf{SO}(n)^m$

such that $\mathsf{det}(\phi) = 0$ is either the whole space or a Zariski subvariety of zero h^m-measure. For almost every $A \in \mathbf{SO}(n)$, $\mathsf{det}(\phi(A, \ldots, A)) \neq 0$ and hence Z_w has zero h^m-measure. Without loss of generality suppose that $U_{w_1} = U_0$. The isometry T_w will not be recurrent if $< \mathsf{Fix}(A_w), U_0 v_0 + \ldots + U_{m-1} v_{m-1} > \neq 0$. For a full measure subset S_1 of $\mathbf{SO}(n)^m$, U_0 has full rank and A_w has a one-dimensional fixed point subspace. Fix an m-tuple $C = (A_0, \ldots, A_{m-1}) \in S_1$ which defines an element $A_w(C)$. Since U_0 has full rank and we may vary v_0 independently of both $(A_0, \ldots, A_{m-1})$ and $(v_1, ..., v_{m-1})$ for a full measure subset of $(v_0, \ldots, v_{m-1}) \in \mathbf{R}^m$, $< \mathsf{Fix}(A_w(C)), U_0(C)v_0 + \ldots + U_{m-1}(C)v_{m-1} > \neq 0$. Thus for a full measure subset of $\mathbf{SE}(n)^m$, T_w is not recurrent. The set of words w is countable and thus almost every piecewise isometry in odd dimensions is not recurrent.

(Note that this proof fails to hold for the one-dimensional case since $\mathbf{O}(1) = \{Id, -Id\}$ and so it is not true that for all words for almost every m-tuple $(A_0, \ldots, A_{m-1})$, $\mathsf{det}(U_{w_1}) \neq 0$). ■

In light of Theorem 1 an immediate corollary for convex partitions is:

Corollary 4 *Let $n > 2$ be odd and $\mathcal{P}$ be a convex partition of $\mathbf{X}$. Then a.e. piecewise isometry T in $\mathbf{SE}(\mathcal{P})$ has no periodic points.*

A consequence of this corollary is that a.e. perturbation of a piecewise isometry T in $\mathbf{SE}(\mathcal{P})$ will destroy any existing periodic points. We illustrate this in Figure 2.

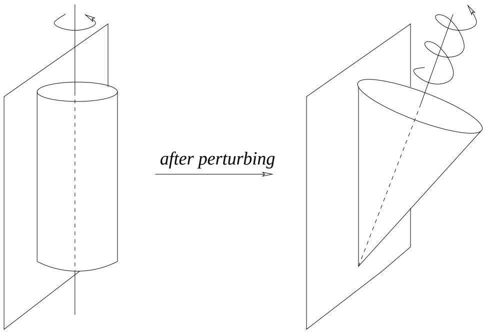


Figure 2: Suppose there exists a $PWI(\mathcal{P})$ on $\mathbb{R}^3$ and the induced isometry T_w corresponding to P_w is recurrent and hence satisfies $< v_w, Fix(A_w) >= 0$. This implies that P_w has cylinder shape. A typical perturbation of T_w leads to linear growth as $< v_w, Fix(A_w) > \neq 0$.

The following result generalises the stability of periodic points stated in Goetz (1996) and (2001).

Theorem 5 *Suppose $T \in PWI(\mathcal{P})$ possesses a periodic point $\mathbf{p}$ with coding w such that $\mathsf{det}(A_w - Id) \neq 0$. Then, $\mathbf{p}$ is stable under perturbation of T. More precisely, for every sufficiently small $\varepsilon > 0$ there is a $\delta > 0$ so that for all perturbations T' of T with*

$dist(T',T) < \delta$ there is a point $\mathbf{p}'$ such that $\|\mathbf{p}-\mathbf{p}'\| < \varepsilon$ and $\mathbf{p}'$ is periodic under T' with coding w.

proof: Let d be the distance between the orbit of $\mathbf{p}$ and the discontinuity set $\mathcal{D}$ and assume $w = w_1 \dots w_k$ where k is the least period of $\mathbf{p}$. For every $1 \leq j \leq k$ the map $e_{w_j \dots w_1} : PWI(\mathcal{P}) \simeq \mathbf{E}(n)^m \to \mathbf{E}(n)$, $e_{w_j \dots w_1}(T) = T_{w_1 \dots w_j}$ is a continuous on the induced isometries $\{(A_i, v_i)\}_{i=0}^{m-1}$. Therefore, given $0 < \varepsilon < d$, we can take $\delta_1 > 0$ such that for all T' with $dist(T,T') < \delta_1$, $T'_{w_j \dots w_1}(B_\varepsilon(\mathbf{p})) \subset T_{w_j \dots w_1}(B_d(\mathbf{p}))$ for all $j = 1, \dots, k$.

Choose $\delta_2 > 0$ so that for T' satisfying $dist(T,T') < \delta_2$ we have $\mathsf{det}(A'_w - Id) \neq 0$ which is possible since $f_w(A_0, \dots, A_{m-1}) = \mathsf{det}(A_w - Id)$ is continuous and by assumption $\mathsf{det}(A_w - Id) \neq 0$.

Let us take $\delta_3 > 0$ such that $dist(T,T') < \delta_3$ and

$$\left\|(A_w - Id)^{-1}.(v_w - v'_w)\right\|_{\mathbb{R}^n} < \varepsilon/2$$

where v_w is the vector part of T_w. This holds by continuity of $g_w(T') = (A_w - Id)^{-1}.(v_w - v'_w)$ in a neighbourhood of T and by noting that $g_w(T) = 0$.

Take $\delta_4 > 0$ such that $dist(T,T') < \delta_4$ and for $h_w(T') = (A_w - Id)^{-1} - (A'_w - Id)^{-1}$,

$$\left\|h_w(T').v'_w\right\|_{\mathbb{R}^n} < \varepsilon/2$$

which also exists by continuity of h_w and the fact that $h_w(T).v'_w = 0$.

Let $\delta = \mathsf{min}\{\delta_1, \delta_2, \delta_3, \delta_4\}$ and T' such that $dist(T,T') < \delta$. Define also $\mathbf{p}' = (A'_w - Id)^{-1}.v'_w$. Thus,

$$\begin{aligned}
\left\|\mathbf{p}-\mathbf{p}'\right\|_{\mathbb{R}^n} &= \left\|(A_w - Id)^{-1}.(v_w - v'_w + v'_w) - (A'_w - Id)^{-1}.v'_w\right\|_{\mathbb{R}^n} \\
&= \left\|h_w(T').v'_w + (A_w - Id)^{-1}.(v_w - v'_w)\right\|_{\mathbb{R}^n} \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

The assertion now follows by noting that $T'_w\mathbf{p}' = \mathbf{p}'$ by definition of $\mathbf{p}'$ and since $\delta < \delta_1$ the orbit of $\mathbf{p}'$ is realised by T'. ■

We note that if $\mathsf{det}(A_w - Id) = 0$ then $\mathsf{Fix}A_w \neq \{0\}$ and therefore it is easy to destroy $\mathsf{Fix}T_w$ by perturbing slightly those parameters $\theta_i = 0$ in the canonical form of A_w.

Corollary 6 *Assume $n \geq 2$ is even. If the defining set of rotations $\{A_i\}_{i=0}^{m-1}$ for $T \in \mathbf{SE}(\mathcal{P})$ is incommensurate then a periodic point for T is stable under perturbations to T.*

Note that by Theorem 2 the previous statement holds for almost every $T \in \mathbf{SE}(\mathcal{P})$. It must be emphasized though, that we cannot guarantee that all periodic points are stable under a given perturbation if there exist an infinite number of periodic points. It may be the case that $\delta = \mathsf{inf}\{\delta_{\mathbf{p}} : \mathbf{p}$ is a periodic point of $T\} = 0$, where $\delta_{\mathbf{p}}$ is calculated as in proof of theorem 5.

4 Discussion: Piecewise Isometries in non-Euclidean spaces

In this paper we have proved results concerning recurrence properties of piecewise isometries defined on Euclidean spaces. Most of our conclusions were based upon group properties of the usual Euclidean action on $\mathbf{R}^n$. It is unlikely that all our results would hold for a general metric space $(\mathbf{X}, d)$.

Nevertheless, some results usually stated in the context of piecewise isometries in Euclidean spaces do remain valid for any metric space. For instance, if the original partition is convex then all cells must be convex as well. Other results related with the measure of cells (Proposition 2.1 and Corollary 2.2 in Goetz (1996)) and the existence of periodic points (Theorem 2.1, Proposition 2.5 and Proposition 2.6 also in Goetz (1996)) hold for any given metric. Theorem 1 in this paper does not depend on whether $\mathbf{X}$ is an Euclidean space.

A standard example of a non-Euclidean metric space is the Poincaré half plane $\mathbf{H} = \{z = x + iy \in \mathbb{C} : \mathsf{Im}(z) \geq 0\}$ endowed with the hyperbolic metric $d_{\mathbf{H}} := \frac{ds}{y^2}$, where ds is the usual Euclidean distance element.

The isometries of $\mathbf{H}$ are given by the group $\mathbf{SL}(2, \mathbb{R})$, consisting of matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \; ad - bc = 1; \; a, b, c, d \in \mathbb{R} \; .$$

To each matrix $A \in \mathbf{SL}(2, \mathbb{R})$ we associate a Möbius transformation $f_A : \mathbf{H} \to \mathbf{H}$ defined by

$$f_A(z) = \frac{az + b}{cz + d} \; .$$

All isometries in $\mathbf{SL}(2, \mathbb{R})$ have bounded orbits since $d_{\mathbf{H}}$ is bounded on $\mathbf{H} \times \mathbf{H}$. From our perspective, it might be equally important to study boundedness and recurrence with respect to the Euclidean metric. Since $f(-d/c) = \infty$ we conclude that f has unbounded orbits whenever $c \neq 0$. However, as in the numerical example below, the recurrence properties of rational cells look surprisingly similar to those of Euclidean piecewise rotations. This might be explained by the fact that isometries on $\mathbf{H}$ can be associated to isometries in the Riemann sphere, which is a compact manifold, and hence behave similarly to piecewise rotations in compact Euclidean spaces.

Let us consider a simple example of an invertible piecewise rotation[2] with two convex atoms defined in $\mathbf{H}$. Take $P_0 := \{z \in \mathbf{H} : \mathsf{Re}(z) < 0\}$, $P_1 := \{z \in \mathbf{H} : \mathsf{Re}(z) \geq 0\}$[3] and let

$$T(z) = \begin{cases} f_0(z) \text{ if } z \in P_0 \\ f_1(z) \text{ if } z \in P_1 \end{cases} \text{ where, } f_0(z) = \frac{z - 0.5}{z + 0.5} \text{ and } f_1(z) = \frac{0.5z - 1}{0.5z + 1} \; .$$

The action of each of the maps f_i on the corresponding atom is depicted in Figure 3.

[2]Rotations in $\mathbf{H}$ are all isometries for which $tr^2 A = (a + d)^2 < 4$ (see Iversen (1992))

[3]Note that we iterate the discontinuity for numerical reasons.

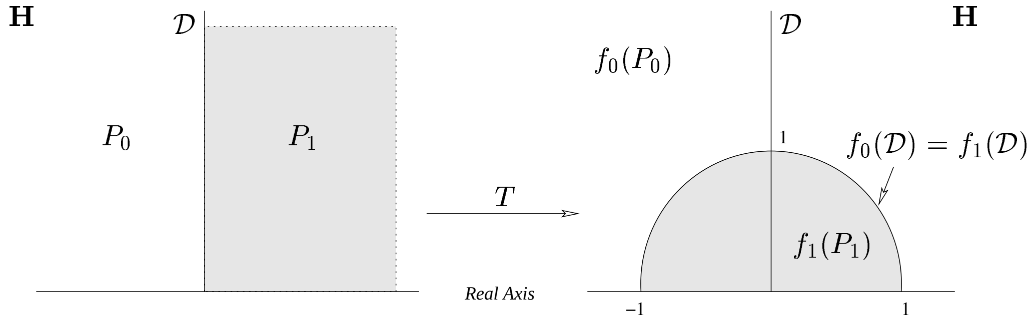


Figure 3: Images of P_0 and P_1 under the action of T.

In order to produce a visualisation of the set of rational cells we compute the preimages of the discontinuity.

The images shown in Figure 4 were produced in the following way. We divide a particular square in the phase space in a grid G of $p \times p$ pixels. For each point in G, we iterated a certain number ($iter$) of times and checked whether its orbit ever got within a given $\varepsilon > 0$ distance of the discontinuity.

Depending on the degree of accuracy and precision needed, these parameters vary in the three graphics presented. Usually, the value of $iter$ is chosen so that the shape of rational cells (regions in white) did not visibly change upon taking a higher number of iterations.

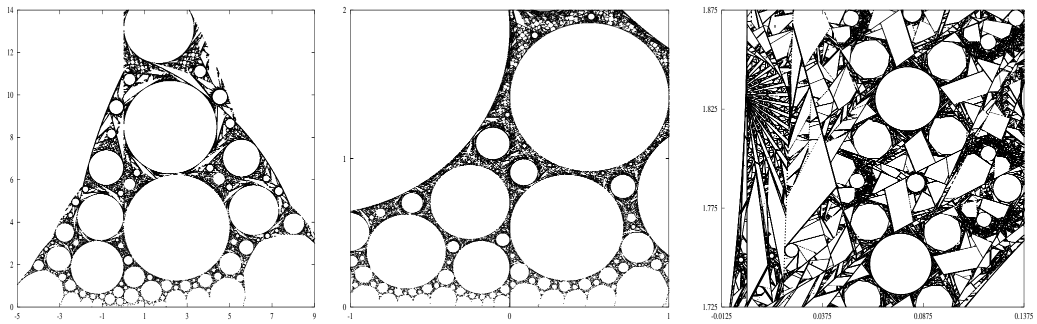


Figure 4: Preimages of the discontinuity under T: (left) $iter = 2500$, $\varepsilon = 10^{-3}$, $p = 1000$ (centre) $iter = 5000$, $\varepsilon = 10^{-4}$, $p = 2000$ (right) $iter = 20000$, $\varepsilon = 10^{-5}$, $p = 5000$.

For a better understanding of the dynamics of T we have also computed the (partial) orbits of three different initial conditions in $\mathbf{H}$. The first two pictures (from left to right) of Figure 5 show orbits of two distinct points inside two different rational cells. The remaining picture was obtained by iterating a point inside the black set.

It is not obvious to us whether results drawn for piecewise rotations in the entire Euclidean plane (see for instance Goetz (1998)) would still hold for the Poincaré half-plane.

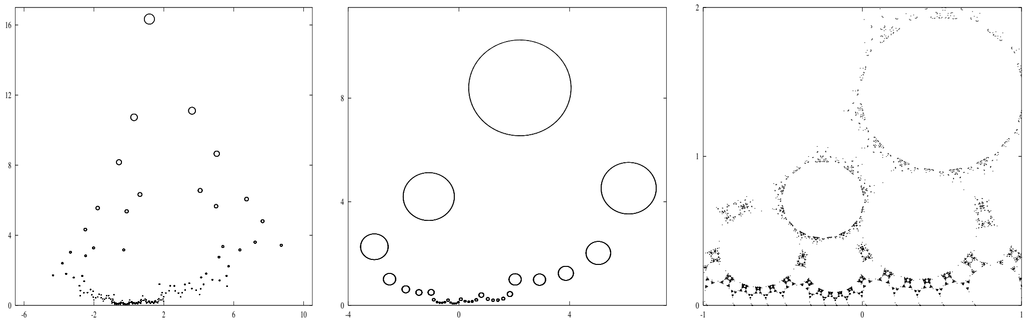


Figure 5: 50000 iterates of: (left) $x = 5.0$, $y = 8.5$ (centre) $x = 0.1$, $y = 0.2$ (right) $x = -0.01$, $y = 0.8$.

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